PREFACE

This book is third in the series "A Textbook of Mathematics For Classes XI-XII." As mentioned earlier, the cardinal question which confronted us was, "Should there be a separate mathematics for different categories of users, viz., students of biology, physics, economics, etc. or would one core course meet the needs of all students in classes XI-XII ?" The feedback received from several teachers and subject specialists tipped the scales in favour of **one core course in mathematics at this stage with situations drawn from various fields of applications.**

A fairly clearcut framework was accordingly evolved by the Mathematics Group at the Council before embarking on preparation of drafts of various units This framework took due cognizance of the pre-requisites of the child, his expected level of attainment, capability for assimilation and other allied factors and determined the mode of introduction of a topic, lines of its development to the optimum level, including the choice of actual life situations and their points of injection. Extensive planning and intensive thinking has, therefore, gone into the base of this book

The first draft was prepared, in keeping with the above guidelines, at a workshop specifically organised for this purpose at the Department of Mathematics, Kurukshetra University, Kurukshetra in which a large number of seasoned teachers, subject experts, besides members of the Mathematics Group at the Council, participated. I record my sincere gratitude to each one of them.

I then undertook to edit and rewrite the materials with valuable assistance of Dr. Ajit Kaur Chilana, Department of Mathematics, University of Delhi ; Dr. Satish Shirali, Centre for Advanced Studies in Mathematics, Panjab University, Chandigarh and Dr. R.P Gupta of DESM. Dr. S.C. Das, Shri G.D. Dhall and Dr A.R. Sahu provided assistance in diverse ways. Shri R. S. Kothari, Dr. S. K. Singh Gautam and Dr. A. R. Sahu provided answers to some of the units. Shri R S. Kothari looked after the preparation of the final press copies. Dr. K C. Madan and Shri Mahendra Shanker edited the Hindi version. I am highly obliged to each one of them for having contributed their mite gladly in carrying the task forward through successive stages.

It is now apt to draw the attention to some of the salient features of the book :

1. The book keeps to the same lucid style as the earlier two in the series. Understanding of concepts and results has been emphasized, often by appealing to the heuristics and even at the cost of rigour at times. This becomes particularly apparent in the treatment of Calculus where the very fundamental concept of limit has been motivated through geometrical and heuristic arguments and where the justifications of the theorems and techniques have been given essentially intuitively.

2. Suitable examples and situational motivation have been provided while introducing a concept.

3. There is an abundance of solved examples to let the student appreciate the wide area of application of a concept.

4. The book is divided into self-contained units. Each unit opens with a preview of what is contained in it and is followed by a list of key concepts to enable the child to quickly review and recall what he has studied in a particular unit.

5. Suggestions for further reading are provided at the end of every unit to let the interested students expand their horizons.

6. The numerous exercises in each unit are intended to place in the hand of the teacher a stock of questions in order that he can meet adequately the needs of the students with differing ability-levels

7. Recall sections, units and exercises have been inserted to highlight the concepts that a child ought to grasp before he goes on to the next section or next unit

8. Miscellaneous exercises have been included at suitable places so that the student can review the materials at appropriate intervals and attain proficiency in the subject.

9. "Why ?" has been added at several places to whet the student's mathematical thinking and to crystallise his ideas. The teacher is, therefore, expected to prod the student to supply the answers rather than simply give the answers himself.

10. Historical anecdotes have been introduced to let the student peep into the process of evolution of a particular concept and see what goes on in the mind of a researcher and a mathematician.

The text-book is only one of the essential tools of teaching-learning process. The importance of the role of the teacher in this process can never be overstated. It is not quite necessary to recall that it is the sacred duty of a teacher to make his subject as interesting and as alive as possible, to generate and sustain the interest of his students and to carry it forward to ever-widening frontiers. I am confident that the teacher will appreciate and join in the philosophy underlying this book, and endeavour to let the student develop a love for and attain high proficiency in the subject.

Finally, I must re-emphasize what I have said to the students in the earlier two books of this series.

The student who is about to study this book on mathematics, should with advantage, imbibe certain '*good*' learning habits from the outset Only then will he find the study of mathematics captivating and rewarding I venture to mention some of the '*good*' learning habits below :

1. **Mathematics is learnt by doing only.** Do not just read your textbook You should always have a pencil and paper with you and '*work through*' the text

2. At places where you come across 'Why ?' in the text, you should strive to discover the answer

3. **Never spend too much time on a particular problem** It is always better to go on to the next and revert to the one that is giving trouble, a little later with a fresh mind

4. The human brain is capable of storing only a limited amount of information What is not used frequently is eliminated from the storage **You will do well, therefore, to make summaries of basic results in every unit and frequently review them.**

Despite our best efforts, it is possible that some unintentional errors might have eluded us. The Council will acknowledge with much gratitude when any of these are pointed out and will welcome the suggestions of its users for improvement in the future editions of this book.

New Delhi
June, 1978

MANMOHAN SINGH ARORA
Editor

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CONTENTS

UNIT XVII

FLOW DIAGRAMS

'Problems' are broken-up into a sequence of logical steps Flow diagrams are introduced as a means of expressing a particular sequencing

17 1 Introduction

Three men come to a river which they have to cross None of them knows how to swim There is no bridge in sight—only two boys fishing in a small boat The men ask the boys if they can help them cross the river However, the boys tell them that the boat is too small—it can carry only one adult or two boys How should the men get across?

A little thought shows that the men can get across by the following scheme ·

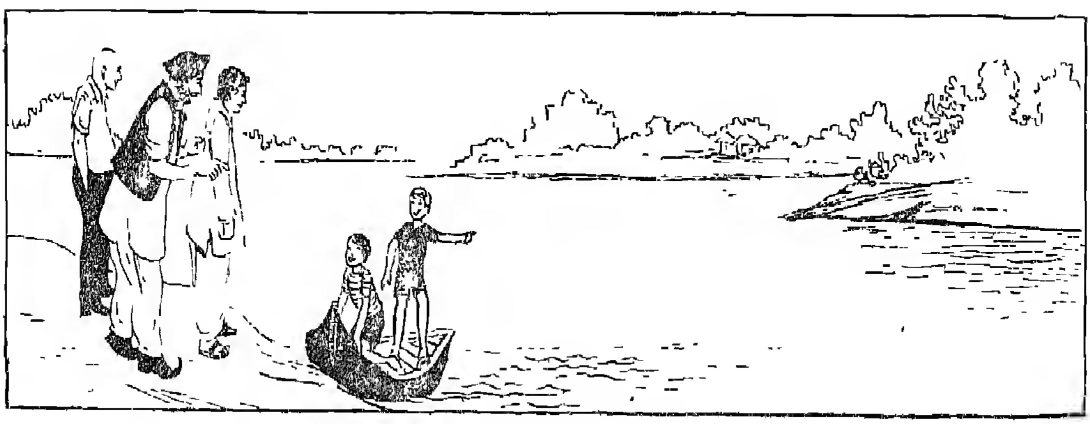


Fig 17.1(i) · Starting Position

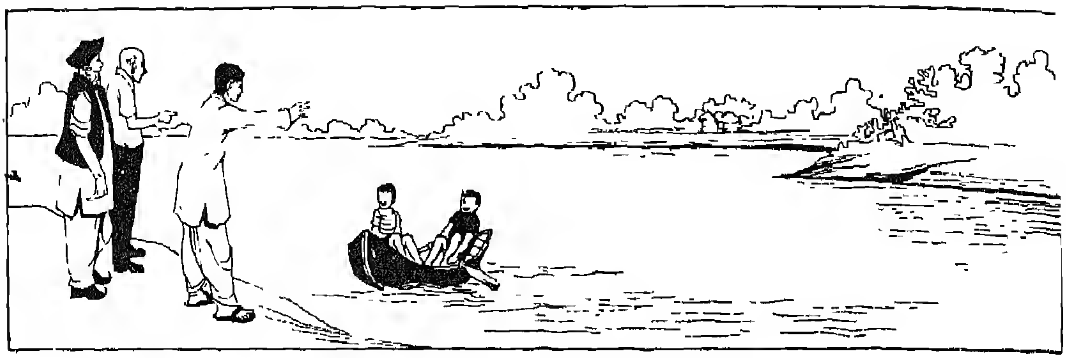


Fig 17.1 (ii) : Two boys row to the other side

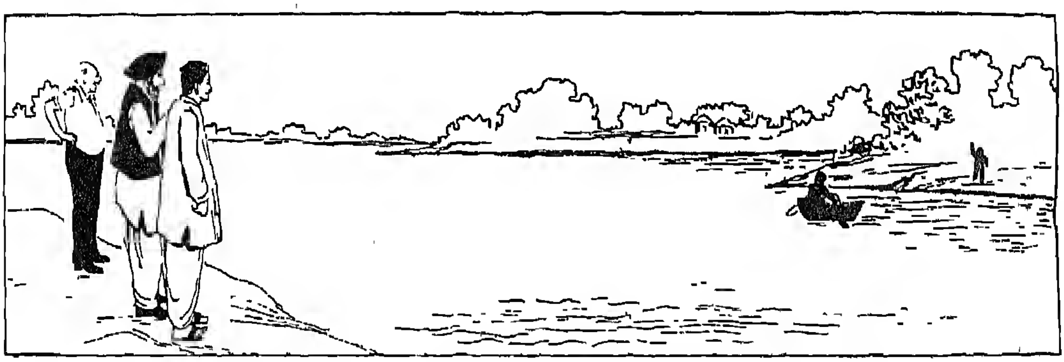


Fig 17 1 (iii) : One boy remains behind The other brings the boat back

Fig. 17 1 (iv) : The boy gets out. One man rows to the other side

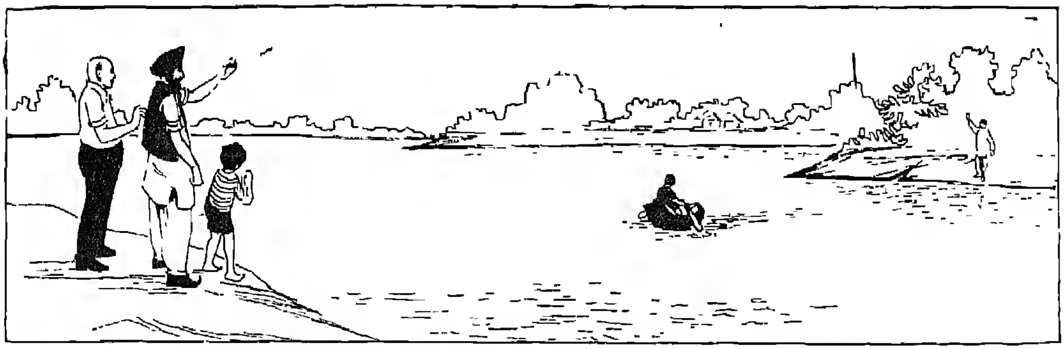


Fig 17.1 (v) · The man gets out. The boy brings the boat back

We are now in a position where one man has crossed to the other side, but the boys and the boat are back · This procedure can be repeated two more times All the three men will then have crossed the river The boys and the boat will again be back on this side

Another method of exhibiting the above scheme is shown in Fig. 17.2.

This method uses what we call a **flow diagram.** In reading a flow diagram, the **'flow'** or **'direction'** of the arrows is followed In other words, **the instructions in the boxes are intended to be carried out sequentially as indicated by the arrows.**

We observe that, in this diagram we have **two types of 'boxes'—rectangles and a rhombus (diamond-shaped). The diamond-shaped box contains a question which calls for making a decision and which has either 'yes' or 'no' for the answer** If the answer is **'yes'**, we go to the top as indicated by the arrow and start again (i e , we repeat the procedure) . If the answer, however, is **'no'**, we are done with the problem. We refer to the diamond-shaped box as the **decision box**

Remark 1 : We again emphasize that a flow diagram is only a method of **exhibiting** a sequence of steps in a solution. It is not a method of **finding** the solution

Remark 2 : Only questions which admit a yes-no answer are included in the decision box.

We are all aware that computers are being increasingly employed in our country to deal with complex problems arising in science, commerce, administration, education, etc The computers have a **'language'** of their own The people who **'talk'** to the computers are known as **programmers.** The programmers often express a problem in a sequence of logical steps before **'feeding'** it to the computer They find that a representation* of a sequence of logical steps, in the form of a flow diagram, is not only helpful but necessary in most cases.

*The programmers also employ some other types of boxes. However, we shall use only the two types mentioned above

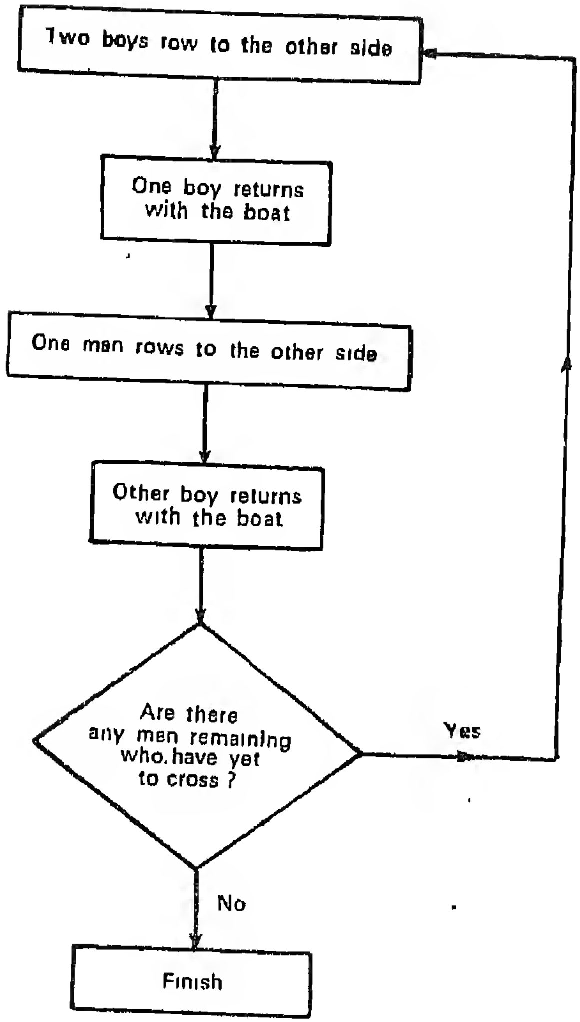


Fig. 17.2

17.2 Reading a Flow Diagram

We follow the direction of the arrows in reading a flow diagram. Let us consider the following examples :

Example 1 : Read the flow diagram in Fig. 17.3. What does it do ?

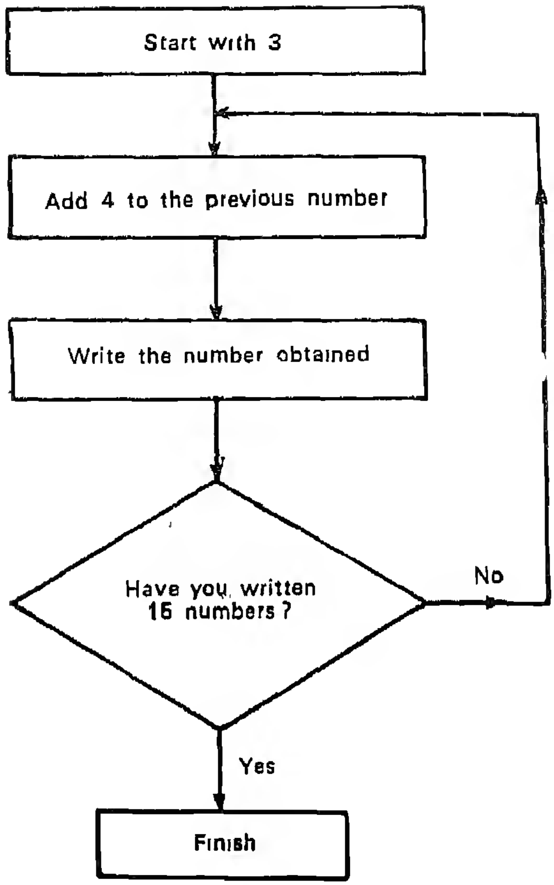


Fig. 17 3

Solution : The flow diagram writes 15 terms of the arithmetic sequence whose first term is 7 and the common difference is 4.

Note that even though we start with the number 3, we do not write it. We begin writing numbers only after we have added 4 to the number 3. This is why the first number, that we write, is 7.

Example 2 : What does the flow diagram in Fig. 17.4 do ?

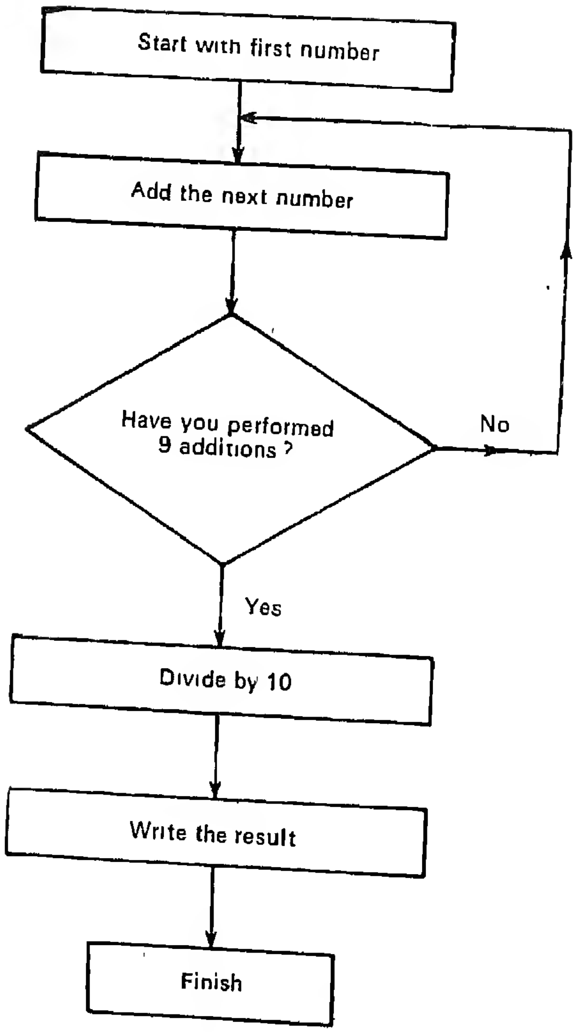


Fig. 17.4

Solution : We are adding ten given numbers and dividing their sum by 10. We are, therefore, finding the mean (or average) of 10 numbers.

17.3 Making a Flow Diagram

We now consider some examples of making flow diagrams

Example 1 : Make a flow diagram to find the real roots of a quadratic equation $ax^2+bx+c=0$, $a \neq 0$, a, b and $c \in R$.

Solution : How shall we proceed? We calculate the discriminant $D=b^2-4ac$ and check if $D \geqslant 0$. If it is, we find the two roots as $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Keeping this procedure in mind, we are able to make a flow diagram as shown in Fig. 17 5.

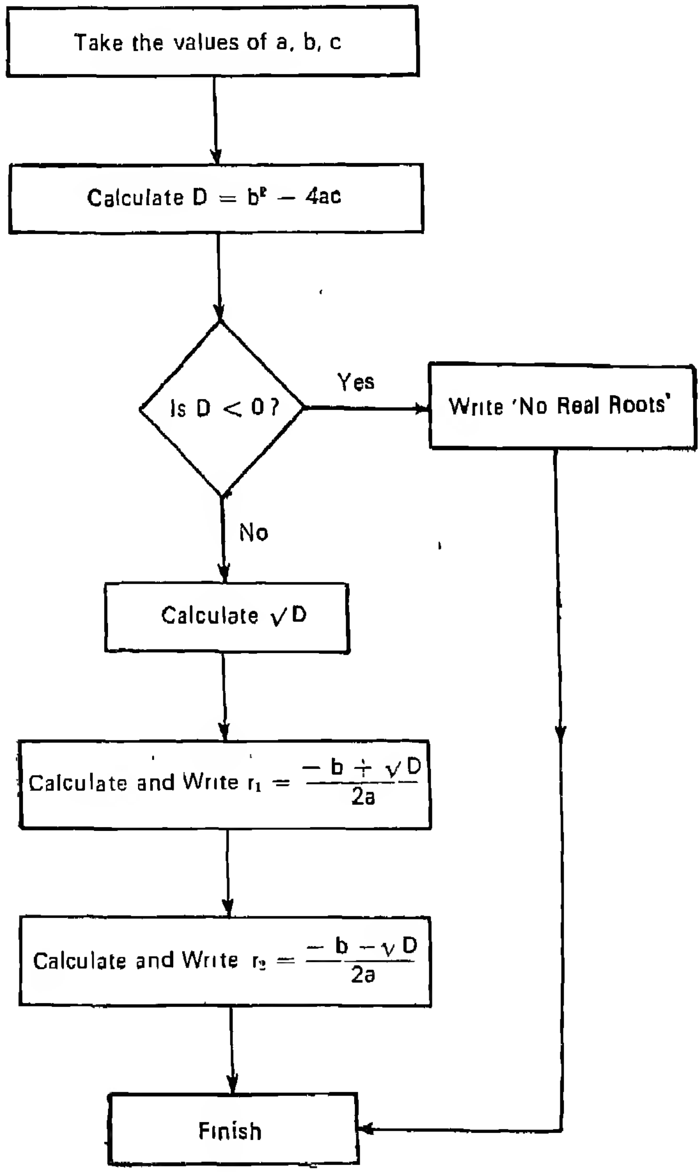


Fig. 17 5

Example 2 : Make a flow diagram to calculate 10!

Solution : How shall we proceed ? We recall that $10! = 1 \times 2 \times 3 \quad . \times 10$ Thus we multiply together the integers from 1 to 10. We, therefore, have a flow diagram as shown in Fig. 17 6

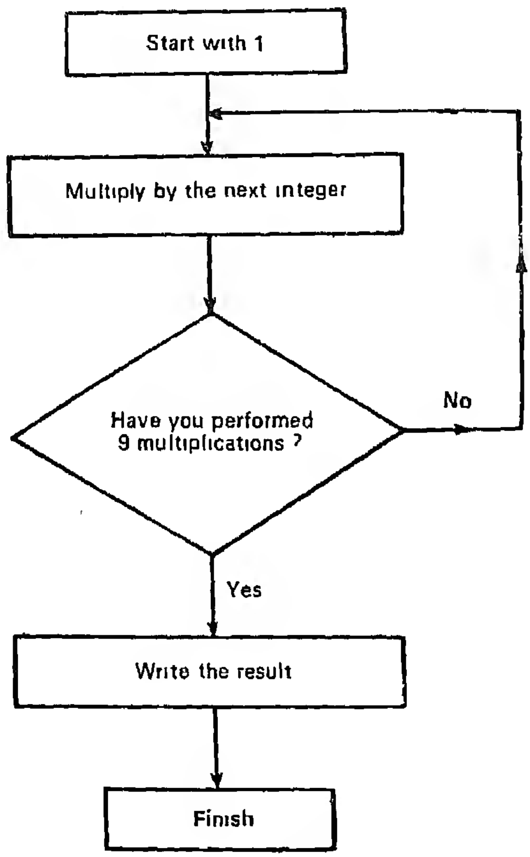


Fig. 17.6

[The reader should note that this flow diagram can be used to calculate $n!$. How? Only, the question in the decision box will have to be suitably worded.]

Example 3 : Make a flow diagram to calculate the compound interest on a certain sum of money, P, at $r\%$ per annum for 12 years.

Solution : We recall that $A=P\left(1+\frac{r}{100}\right)^n$ where A,P, r and n have their usual meanings. Here $n=12$. Thus, $A=P\left(1+\frac{r}{100}\right)^{12}$. We need to calculate the interest,

I, which is equal to the difference of this amount A and the principal, P, i e.,

$$I = P\left(1+\frac{r}{100}\right)^{12} - P$$

$$= P\left[\left(1+\frac{r}{100}\right)^{12} - 1\right]$$

A required flow diagram is given in Fig 17 7

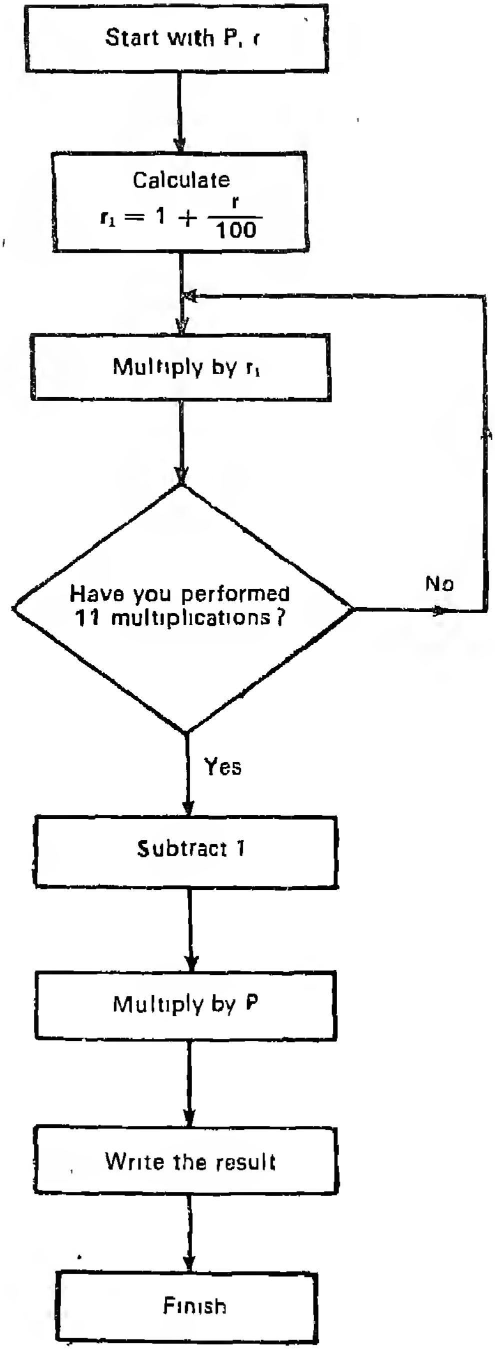


Fig. 17.7

In the last flow diagram, we have calculated $P\left[\left(1+\frac{r}{100}\right)^{12}-1\right]$. Alternatively, we could have calculated $P\left(1+\frac{r}{100}\right)^{12}-P$. In that case, a flow diagram could be as shown in Fig. 17.8.

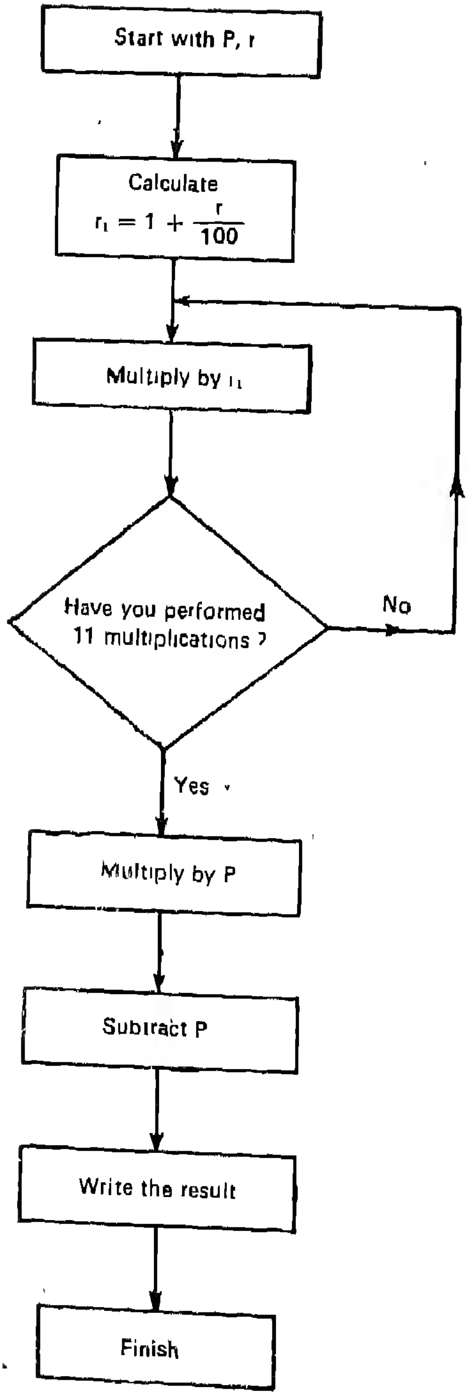


Fig. 17.8

It should be remarked that it is not uncommon for two persons to make two different flow diagrams for the same problem.

Example 4 : Make a flow diagram to calculate the sum of squares of first n natural numbers

Solution : We want to calculate $1^2+2^2+\ldots+n^2$ Thus starting with 1, we square each integer, and keep performing additions until we have squared the integer n and added it to the sum already obtained. A flow diagram is given in Fig 17 9

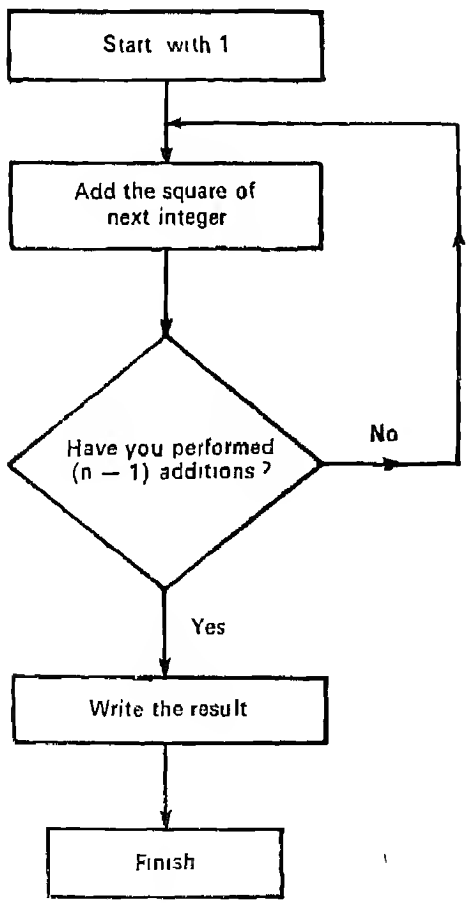
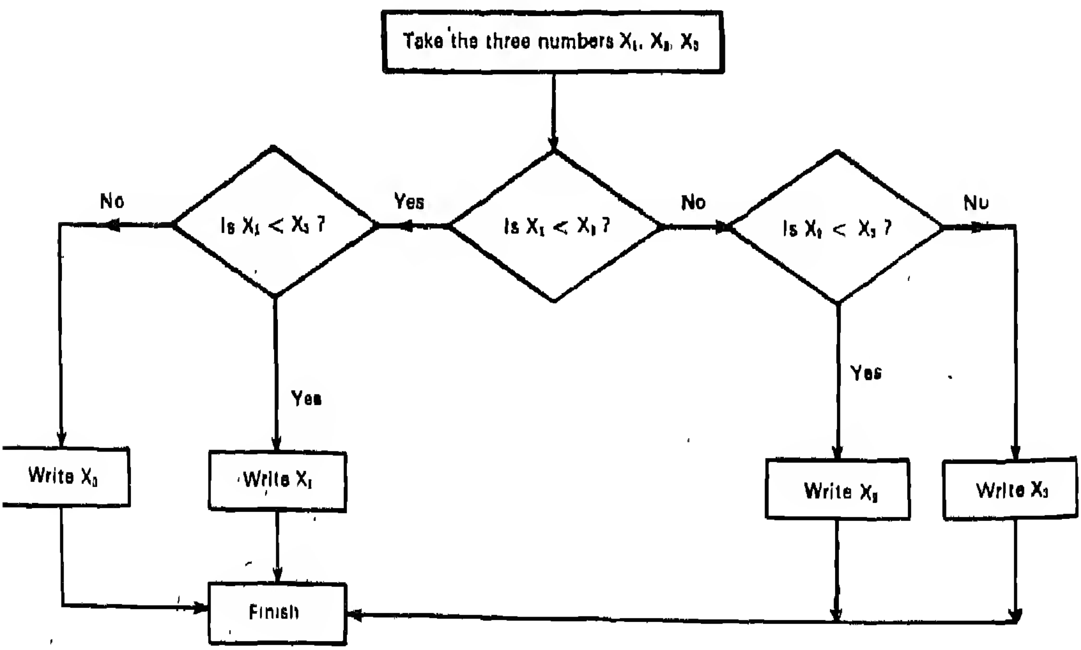


Fig. 17 9

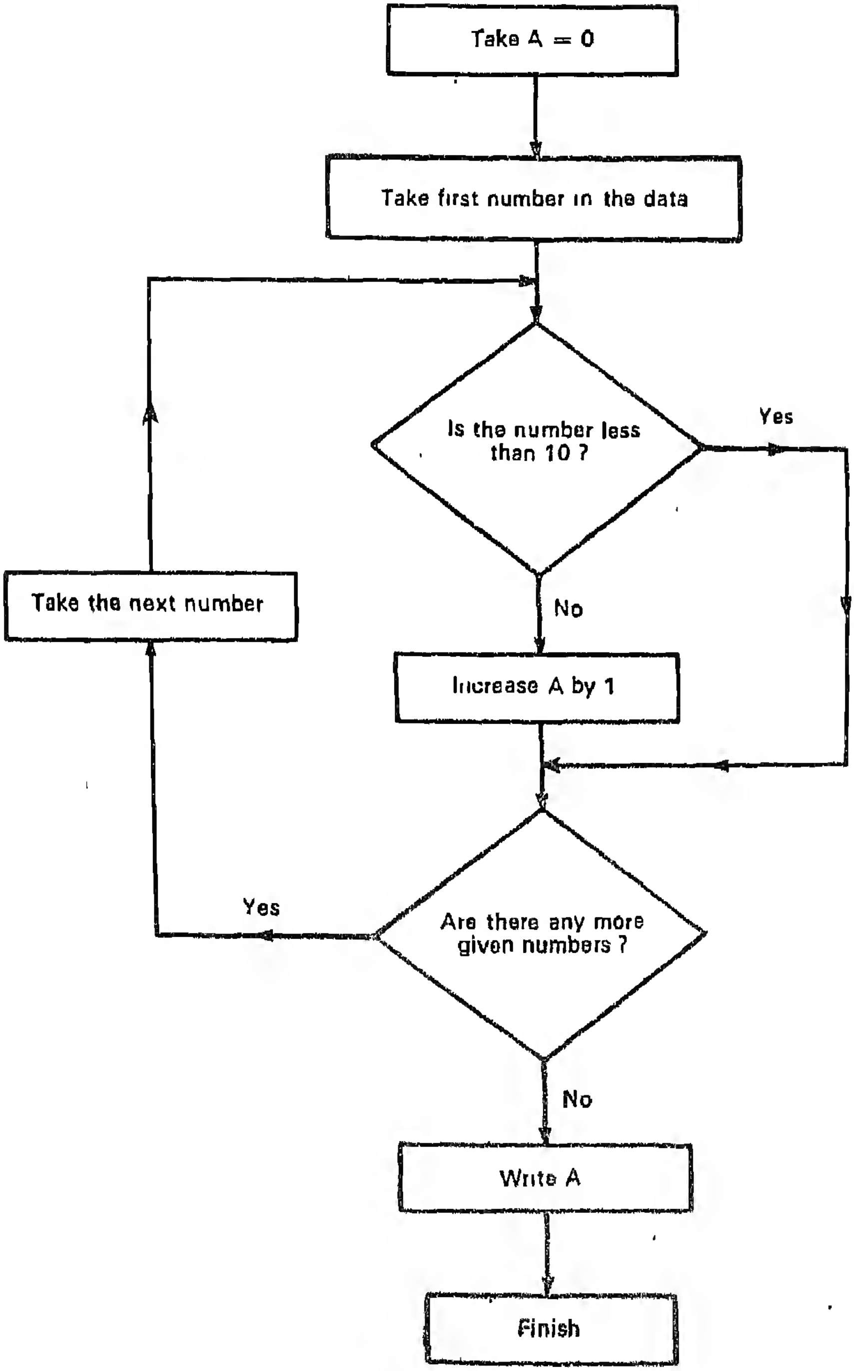
EXERCISE 17.1

Make a flow diagram for each of the following problems. **Recall that different flow diagrams may be possible for the same problem.**

1. Write the first 20 terms of an arithmetic sequence, whose first term is -8 and common difference is 2.
2. Calculate the sum of first 100 natural numbers, without using the formula.
3. Calculate the value of $2^{15}-15$.
4. Calculate the mean (or average) of n given numbers. Perform only one addition at a time.
5. Write the first 15 terms of a geometric sequence whose first term is $\frac{3}{2}$ and common ratio is -2.
6. Calculate r^n and hence, the sum of n terms of a geometric sequence whose first term is 'a' and common ratio is 'r' Assume $r>1$.
7. Write the 15th to 20th powers of 4.
8. What does the following flow diagram do ?



9. Read the following flow diagram. What does it do ?



17.4 Key Concepts

Flow diagram	Decision box

17.5 Suggestions for Further Reading

The reader is referred to any simple book on computer programming, for instance,

[1] D.G Moursund **How Computers do it.**
Wadsworth Publishing Company, Inc., California. (U S A). 1969

UNIT XVIII

WHAT IS CALCULUS ABOUT ?

In this unit, we apprise the reader about the problems that are studied in Calculus In particular, the problem of speed is discussed and is used to provide a motivation for learning Calculus. Derivative as limit of difference quotient is introduced

18.1 Introduction

The word **'Calculus'** is a **Latin** word, which means a **'pebble'** or a **'small stone'**. In ancient times, pebbles were used as aids for calculations. In fact, the word **'calculate'** is also derived from the same Latin word **'Calculus'** When we speak of **Calculus** today, however, we mean **the theory** and **techniques that we study in, what are called, Differential and Integral Calculus.**

If we examine the history of development of mathematics and the history of development of science and technology in any culture, we find that the two are closely linked. Of course, this does not come as a surprise to us. For, when the scientists find the existing tools and techniques inadequate to *'solve'* the problems of the day, they try to *'devise'* and *'create'* new tools and techniques. It was this *'search'* that led to the development of, what we call, **plane geometry**—the *'search'* for and the necessity of finding the areas and perimeters of land and determining when two figures are *'congruent'* or *'similar'*. Again, it was this *'search'* that led to the creation of **trigonometry**—the *'search'* for the techniques to find the sizes of heavenly bodies and distances between them.

18.2 What is Calculus About ?

Calculus begins with a simple question, "**What is the speed of a moving body at any particular instant of time and how do we calculate it ?**" This question plagued the

physicists and astronomers of the 17th century, who were very much concerned with the problems of motion. Another related problem in the study of motion is to find the direction of motion of a moving body at any point of its (curvilinear) path. It turns out that this is essentially the problem of finding the direction of tangent to the path

Besides providing satisfactory answers to the above questions, the techniques of Calculus also yield solutions to several other problems We do not intend to catalogue these various types of problems here To do so will be meaningless for the reader at this stage. Our strategy therefore, will be to point them out when and where we meet them.

Evolving out of the attempts to answer these simple questions about speed, Calculus has made a tremendous impact, not only in science and technology, but also in such fields as economics, business, etc. New branches in mathematics, that use Calculus, have appeared. The study of Calculus is, therefore, a must for any one wanting to be '*literate*' in mathematics.

18.3 The Problem of Speed

We now turn our attention to the **problem of speed** Let us first recall what we mean by 'speed'. Speed is the rate of change of distance with respect to time

How do we calculate the speed of a moving body ? If, for instance, it takes us 2 hours to travel 120 km between two points, we say our speed is $\frac{120}{2}$ i e , 60 km/hour In other words, **if a body travels s_1 units of distance in t_1 units of time and s_2 in t_2 units ($t_2 > t_1$), then the average speed of this body from time t_1 to t_2 is calculated as**

$$\frac{s_2 - s_1}{t_2 - t_1}$$

This is the concept of speed that we have learnt However, the body may not move with the same speed throughout the time interval Have you ever looked at the speedometer of a moving car or bus or tractor ? Do you think the speed that you observe is the average speed ? No! It is the speed at the particular **instant** at which you look at the speedometer **How do we calculate the speed of a moving body at a particular instant?** In other words, how do we calculate the **instantaneous speed** of a moving body?

We are now making a distinction between **interval** and **instant** In calculating the **average speed**, we consider the speed of a body over an **interval** of time. This interval might be large or might be small. In fact, it might even be very small. However, when we speak of an **instant**, we envisage it to be so small, so very small, that no time elapses. We shoult point out that like the notion of a '*point*' in Geometry, the notion of an '*instant*' in Calculus is also a mathematical abstraction

The notion of speed at an instant presents some difficulties How do we calculate the speed at an instant? The distance travelled in an instant is 0 and so is the time elapsed If we apply the method that we have learnt, we are faced with the meaningless quantity $\frac{0}{0}$. We must, therefore, evolve some other method of calculating instantaneous speed of a body moving with a variable speed

Let us consider the motion of a body Let us suppose that the body travels according to $s=4.9\,t^2$, where s is the distance travelled (in metres) and t is the time (in seconds)

We construct a table of distances travelled by the body at the end of 1 second, 2 seconds, 3 seconds, etc

Time elapsed (in seconds)	0	1	2	3	4	5
Distance travelled (in metres)	0	4 9	19 6	44 1	78.4	122 5

Certainly, you can calculate the average speed of this body during 0 to 1 seconds, 1 to 2 seconds, 2 to 3 seconds, etc We observe that the average speed varies from one interval of time to the next We thus conclude that **the body is moving with a variable speed**

How about the speed at the precise instant, say, 3 ? As remarked earlier, it is certainly **not** the average of the speeds from 0 to 3 seconds, or from 1 to 3 seconds, or from 2 to 3 seconds

Let us now see what happens if we calculate the averages over smaller and smaller intervals with 3 as the end-point But why smaller and smaller intervals? It is because, **although the body is moving with a variable speed, we would expect that smaller the interval of time, lesser the variation in its speed.** Let us see if this is indeed true

We again construct a table of distances against times elapsed

Time elapsed (in seconds)	2	2.5	2.6	2 7	2 8	2 85	2.9	2 95
Distance travelled (in metres)	19.6	30 625	33 124	35 721	38 416	39.80025	41 209	42 64225

Time elapsed (in seconds)	2.96	2.97	2.98	2.99	2.995	3
Distance travelled (in metres)	42.93184	43.22241	43.51396	43.80649	43.9531225	44.1

We calculate the averages over smaller and smaller intervals with 3 as the end-point We record these in the table below ·

Time interval,	0 to 3	1 to 3	2 to 3	2.5 to 3	2.6 to 3	2.7 to 3	2.8 to 3	2.85 to 3	2.9 to 3
Length of the time interval (in seconds)	3	2	1	0.5	0.4	0.3	0.2	0.15	0.1
Average speed (m/sec)	14.7	19.6	24.5	26.95	27.44	27.93	28.42	28.665	28.91

Time interval	2.95 to 3	2.96 to 3	2.97 to 3	2.98 to 3	2.99 to 3	2.995 to 3
Length of the time interval (in seconds)	0.05	0.04	0.03	0.02	0.01	0.005
Average speed (m/sec)	29.155	29.204	29.253	29.302	29.351	29.3755

Let us examine the average speeds They are .. , 26.95, 27.44, 27.93, 28.42, 28.665, 28.91, 29.155, 29.204, 29.253, 29.302, 29.351, 29.3755,... **Do they seem to come closer and closer to some fixed number?** Indeed, they do Can you guess this number? 29.4 seems to be a '*fair*' guess. Thus, we observe that **as the length of the time intervals becomes smaller and smaller, the average speed becomes closer and closer to a fixed number.** Mathematically, we say that as the length of the time intervals **approaches** zero, the average speed **approaches** a fixed number, called the **limit** of the average speeds.

[The reader is advised to verify that if the length of the time intervals approaches zero **from the other side**, namely, through the time intervals (3 to 3 5), (3 to 3 4), (3 to 3 3), (3 to 3 2), (3 to 3.15), (3 to 3.1), (3 to 3 05), (3 to 3 04), (3 to 3 03), (3 to 3 02), (3 to 3 01), (3 to 3 005), etc —**again the average speed approaches some fixed number.** Can you guess what this number is?]

It, therefore, seems reasonable to take this limit as the speed at the precise instant in question. For our example, we will say that the speed of the body at the instant 3 is 29 4 m/sec

This method of finding instantaneous speed is obviously very tedious As we shall see, **Calculus provides us with an elegant way of finding these limits.**

EXERCISE 18.1

1 The distance, s, travelled by a body in time, t, is given by the formula $s = t^2$ Calculate

(i) The average speed of the body during the first 4 seconds

(ii) The average speed during the fourth second

(iii) The speed at the instant 4. (Try to make a '*fair*' guess)

18.4 Difference Quotients

Let us again refer to the example studied in Section 18 3. A body is moving according to

$$s = 4\ 9t^2 \tag{1}$$

We observe that **for each (non-negative) value of t, there is a (associated) value of s,** which is given by the above formula. [In fact, this is how we calculated the table of values of t and s in Section 18.3.] Thus the formula defines what we call a **real function.** We recall that *a real function is a way of associating with every number of a non-empty set A $(A \subset R)$ a unique number of a set $B(B \subset R)$*, where R denotes the set of real numbers

We shall concern ourselves with **real functions** in Calculus Henceforth, we shall refer to them, simply, as **functions.**

Let us use the function notation and rewrite (1) as

$$f(t) = 4\ 9t^2 \tag{2}$$

What is $f(2)$? Recall, it is the value of the function at $t=2$ Thus, $f(2) = 4\ 9(2)^2 = 19\ 6$ What does $f(2)$ represent ?

What is $f(3)$? What does $f(3)$ represent?

What is $\frac{f(2)-f(3)}{2-3}$? It is $\frac{19.6-44.1}{2-3}$, *i.e.*, 24.5 What does it represent? It represents* the **average speed of the body during the third second, i.e., during the time interval (2 to 3 seconds).**

What does $\frac{f(2.85)-f(3)}{2.85-3}$ represent? It represents the average speed of the body during the time interval (2.85 to 3 seconds).

So here is a convenient way of representing average speeds, using the function notation. The expressions $\frac{f(2)-f(3)}{2-3}$, $\frac{f(2.85)-f(3)}{2.85-3}$ are called **'difference quotients,** since they are **quotients of differences**

We recall that to calculate the speed at the instant 3, we had to compute a whole lot of average speeds, averages taken over smaller and smaller intervals with 3 as the end-point We observed that these averages approached a limit and this limit was taken to be the speed at the instant 3. In terms of difference quotients, we computed

$$\frac{f(2)-f(3)}{2-3}, \quad \frac{f(2.5)-f(3)}{2.5-3}, \quad \frac{f(2.6)-f(3)}{2.6-3}, \text{ etc} \tag{3}$$

Also, we advised the reader to calculate

$$\frac{f(3.5)-f(3)}{3.5-3}, \quad \frac{f(3.4)-f(3)}{3.4-3}, \quad \frac{f(3.3)-f(3)}{3.3-3}, \text{ etc.} \tag{4}$$

It is the limit of these difference quotients that we are interested in.

We now introduce a slightly more convenient notation to be able to talk about various difference quotients, all at the same time

Let us denote** by $\triangle t$ (read as 'delta t') **the difference in time t.** For instance, let

$$\triangle t = 2-3$$

Then, $$2 = 3+\triangle t$$

We can now write the first difference quotient in (3) as $\frac{f(3+\triangle t)-f(3)}{\triangle t}$ We note that $\triangle t<0$.

---

*A more natural way to represent the average speed would, perhaps, be $\frac{f(3)-f(2)}{3-2}$. However, mathematically, the two are equivalent For reasons to become clear later, we use $\frac{f(2)-f(3)}{2-3}$ to represent the average speed

**$\triangle t$ is not the product of $\triangle$ and t It is simply a convenient symbol to denote differences in t, $\triangle$ being the Greek letter for D.

Now let $\triangle t = 2.5-3$

Then, $2.5=3+\triangle t$

We can, therefore, also write the second difference quotient in (3) as $\frac{f(3+\triangle t)-f(3)}{\triangle t}$. Again, we note that $\triangle t<0$

We see that in this manner, the difference quotients in (3) can *all* be written as

$$\frac{f(3+\triangle t)-f(3)}{\triangle t}$$

with $\triangle t$ taking different (negative) values.

Now let $\triangle t=3.5-3$

Then, $3.5=3+\triangle t$

We can, therefore, write the first difference quotient in (4) as $\frac{f(3+\triangle t)-f(3)}{\triangle t}$.
We note that $\triangle t>0$

Again, let $\triangle t=3.4-3$

Then, $3.4=3+\triangle t$

The seco difference quotient in (4) can, therefore, also be written as

$$\frac{f(3+\triangle t)-f(3)}{\triangle t}$$

Again, we note that $\triangle t>0$. Thus, again the difference quotients in (4) can *all* be written as

$$\frac{f(3+\triangle t)-f(3)}{\triangle t}$$

with $\triangle t$ taking different (positive) values.

With this $\triangle t$ notation, therefore, it is possible to write* each difference quotient as

$$\frac{f(3+\triangle t)-f(3)}{\triangle t} \tag{5}$$

where $\triangle t$ takes different positive or negative values

Further, it is the limit of these difference quotients in (5), as $\triangle t$ approaches zero, that we take as the speed of the moving body at the (precise) instant 3. Mathematically we **write the speed at instant 3 as $\lim_{\triangle t\to 0}\frac{f(3+\triangle t)-f(3)}{\triangle t}$, where 'Lim' is an abbreviation for the word 'limit' and $\lim_{\triangle t\to 0}$ means that the limit is to be taken as $\triangle t$ approaches zero.**

*In order that we may write all the difference quotients in form (5), we wrote the average speeds in (3) with negative denominators.

Remark : The phrase '**Δt approaches zero**' is a mathematical idealization. It simply means that we are considering difference quotients over smaller and smaller intervals. However, **we never consider Δt equal to zero.**

We will see in Section 18 5 that techniques of Calculus provide us with an elegant method of finding such limits without our having to go through the tedious calculations of Section 18 3.

18.5 Limits of Difference Quotients : Derivatives

Let us now see if we can find

$$\lim_{\Delta t \to 0} \frac{f(3+\Delta t)-f(3)}{\Delta t}$$

by some method more convenient than before.

Certainly, we know how to calculate $f(3)$, $f(3+\Delta t)$, etc Again taking $f(t)= 4 9\ t^2$, we note that

$$f(3+\Delta t)=4.9\ (3+\Delta t)^2=44\ 1+29.4(\Delta t)+4.9(\Delta t)^2$$

And, $\quad f(3)=44\ 1$

Thus, $\quad f(3+\Delta t)-f(3)=29.4(\Delta t)+4\ 9(\Delta t)^2 \quad (1)$

Dividing both sides of (1) by Δt, we have

$$\frac{f(3+\Delta t)-f(3)}{\Delta t}=29.4+4.9(\Delta t) \quad (2)$$

As $\Delta t \to 0$, $4.9(\Delta t)$ approaches zero and, therefore, R.H.S. in (2) approaches 29.4. Thus, the limit of the average speed, as Δt approaches zero, is 29.4. In other words, the speed at the instant $t=3$ is 29 4 m/sec.

Example : Let us now use this method and find the speed at the instant $t=4$ when $f(t)=4.9t^2$

To find the speed at the instant 4, we must compute

$$\lim_{\Delta t \to 0} \frac{f(4+\Delta t)-f(4)}{\Delta t}$$

Now, $\quad f(4+\Delta t)=4\ 9\ (4+\Delta t)^2=78\ 4+39.2(\Delta t)+4.9(\Delta t)^2$

And, $\quad f(4)=78.4$

Thus, $\quad f(4+\Delta t)-f(4)=39.2(\Delta t)+4.9(\Delta t)^2 \quad (3)$

Dividing both sides of (3) by Δt, we have

$$\frac{f(4+\Delta t)-f(4)}{\Delta t}=39.2+4.9(\Delta t) \quad (4)$$

As $\triangle t \to 0$, $4.9(\triangle t) \to 0$ and, therefore, R.H.S in (4) approaches 39.2 Thus, the speed at the instant $t=4$ is 39.2 m/sec

Can we generalize this method ? In other words, given f (t)=4.9 t², can we find the speed at some general instant t ? How shall we proceed ? To find the speed at instant t, we must compute

$$\lim_{\triangle t \to 0} \frac{f(t+\triangle t)-f(t)}{\triangle t}$$

Now, $\quad f(t+\triangle t)=4.9\ (t+\triangle t)^2=4.9t^2+9.8t(\triangle t)+4.9(\triangle t)^2$

Thus, $\quad f(t+\triangle t)-f(t)=4.9t^2+9.8t\ (\triangle t)+4.9(\triangle t)^2-4.9t^2$

i e, $\quad f(t+\triangle t)-f(t)=9.8t(\triangle t)+4.9(\triangle t)^2 \qquad (5)$

Hence, $\quad \dfrac{f(t+\triangle t)-f(t)}{\triangle t}=9.8t+4.9(\triangle t) \qquad (6)$

As $\triangle t \to 0$, $4.9(\triangle t) \to 0$ and, therefore, R H S in (6) approaches $9.8t$ Thus, speed at instant t is $9.8t$ m/sec

[The reader is advised to verify that for $t=3$, this indeed works out to 29.4 m/sec, and for $t=4$, this works out to 39.2 m/sec]

Thus, **for each positive value of t, there is an associated value of 9.8t, which gives the speed at instant t** This function is called the **derived function** or **derivative function** or simply, **derivative of f** and is **denoted by f′** (read as **'f-dash'**). In other words, if

$$f(t)=4.9t^2,$$

$$f'(t)=\lim_{\triangle t \to 0} \frac{f(t+\triangle t)-f(t)}{\triangle t}=9.8t$$

The values of $\lim_{\triangle t \to 0} \dfrac{f(t+\triangle t)-f(t)}{\triangle t}$ for $t=3,4$, etc. are called **derivatives of f at $t=3,4$,** etc.

This method of finding limits of difference quotients is very powerful indeed. For one, it bypasses all those tedious and messy calculations Secondly, we get an exact value of the limit which we can 'almost' read off from the general expression for difference quotients. And finally, we need not work with specific values of t. As we have seen, we can find the limit for 'general' t and afterwards substitute any value of t that is desired. We, sometimes, refer to this method of finding derivatives as the **delta method.**

If, instead of $f(t)=4.9t^2$, we use the notation $s=4.9t^2$, then $\triangle s$ can be used* to denote $f(t+\triangle t)-f(t)$ and we have

$$\triangle s=f(t+\triangle t)-f(t) \qquad (7)$$

*Again, $\triangle s$ is not the product of $\triangle$ and s. It is simply a convenient symbol to denote differences in s.

Then, $\lim\limits_{\triangle t \to 0} \dfrac{f(t+\triangle t)-f(t)}{\triangle t}$ is the same as $\lim\limits_{\triangle t \to 0} \dfrac{\triangle s}{\triangle t}$, which can further abbreviated as $\dfrac{ds}{dt}$ or $\dot{s}$.

How do we calculate $\triangle s$? By the definition in (7),

$$f(t+\triangle t)=f(t)+\triangle s=s+\triangle s$$

Thus, we first calculate $s+\triangle s$ by replacing t by $t+\triangle t$ in the R H S of $s=4.9$ and then subtract s from the result

The symbol $\dfrac{ds}{dt}$ was introduced by the German mathematician, **Gottfried Wilhelm Leibniz (1646-1716)** This symbol should not be mistaken for the quotient of ds and dt In fact, nowhere have we used or given any meaning to the symbol ds or dt The symbol $\dot{s}$ was introduced by the English scientist, **Isaac Newton (1642-1727)** We, however, will avoid the use of the symbol $\dot{s}$ in this book.

We now find derivatives of some simple functions. Unless otherwise specified, we will take the domain to be the set R of real numbers.

Example 1 : Given $f(x)=x^2$, find $f'(x)$ by the delta method

Solution : We have

$$f(x)=x^2$$

Thus, $f(x+\triangle x)-f(x) = (x+\triangle x)^2-x^2=2x(\triangle x)+(\triangle x)^2$

And, $\dfrac{f(x+\triangle x)-f(x)}{\triangle x}=2x+\triangle x$

Hence, $f'(x)=\lim\limits_{\triangle x\to 0} \dfrac{f(x+\triangle x)-f(x)}{\triangle x}=\lim\limits_{\triangle x\to 0}(2x+\triangle x)=2x$

Example 2 : Given $f(x)=ax^2$, where 'a' is a constant, find $f'(x)$ by the delta method. Hence, find $f'(2)$.

Solution : We have

$$f(x)=ax^2$$

Thus, $f(x+\triangle x)-f(x) = a(x+\triangle x)^2-ax^2=2ax(\triangle x)+a(\triangle x)^2$

And, $\dfrac{f(x+\triangle x)-f(x)}{\triangle x}=2ax+a(\triangle x)$

Hence, $f'(x)=\lim\limits_{\triangle x\to 0} \dfrac{f(x+\triangle x)-f(x)}{\triangle x}=\lim\limits_{\triangle x\to 0}[2ax+a(\triangle x)]$

Now, as $\triangle x \to 0$, $a(\triangle x) \to 0$

Thus, $f'(x) = \lim_{\triangle x \to 0} \dfrac{f(x+\triangle x)-f(x)}{\triangle x} = 2ax$

And, $f'(2) = \text{Value of } f'(x) \text{ at } (x=2)$

$= 4a$

Remark : We observe that the derivative of a scalar multiple of x^2 is the scalar multiple of the derivative of x^2.

Example 3 · Let $y = ax^2+3$, where 'a' is a constant. Find $\dfrac{dy}{dx}$ by the delta method. Hence, find $\dfrac{dy}{dx}$ at $x=-1$

Solution : We first calculate $y+\triangle y$ by replacing x by $x+\triangle x$ in the R H S of $y=ax^2+3$. We have

$$y+\triangle y = a(x+\triangle x)^2+3$$

Since, $\triangle y = (y+\triangle y)-y$, it follows that

$$\triangle y = a(x+\triangle x)^2+3-(ax^2+3)$$

i.e., $\triangle y = 2ax(\triangle x)+a(\triangle x)^2$

Hence, $\dfrac{\triangle y}{\triangle x} = 2ax+a(\triangle x)$

Taking the limit of $\dfrac{\triangle y}{\triangle x}$ as $\triangle x \to 0$, we have

$$\frac{dy}{dx} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \to 0} [2ax+a(\triangle x)] = 2ax$$

Now $\dfrac{dy}{dx}$ at $(x=-1) = \text{Value of } \dfrac{dy}{dx}$ at $(x=-1)$

$= -2a$

It is often more convenient to denote the value of $\dfrac{dy}{dx}$ at $(x=-1)$ by the symbol

$$\left.\frac{dy}{dx}\right]_{x=-1} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=-1}$$

EXERCISE 18.2

1. Given $f(x) = 2x^2$, compute

   (a) $\dfrac{f(3.8)-f(4)}{3.8-4}$ (b) $\dfrac{f(3.9)-f(4)}{3.9-4}$

   (c) $\dfrac{f(4.1)-f(4)}{4.1-4}$ (d) $\dfrac{f(4.3)-f(4)}{4.3-4}$

2. The distance s of a body moving in a straight line, in time t, is given by the formula $s = t^2$, $(t \geqslant 0)$. Determine the speed at $t = 1$, $t = 3$, $t = 4$. Is the body moving with uniform speed? Give reason for your answer.

3. An object, near the surface of the moon, falling freely under gravity (of the moon) and starting from rest, falls according to the formula $s = 0.8\,t^2$, $(t \geqslant 0)$, where s is measured in metres and t in seconds. Use the delta method to determine its speed at the end of the 2nd second.

4. Given $f(x) = \frac{1}{2}x - 3$, find $f'(x)$. Hence calculate $f'(0)$, $f'(-\frac{1}{2})$, $f'(2)$.

5. Given $g(x) = -2x^2$, use the delta method to find $g'(x)$. Hence find $g'(0)$, $g'(-1)$ and $g'\left(\dfrac{3}{2}\right)$.

6. Given $h(r) = \pi r^2$, use the delta method to find $h'(r)$. Hence find $h'\left(\dfrac{5}{2}\right)$ and $h'(\pi)$.

7. Let $y = 2 - 3x^2$. Use the delta method to find $\left.\dfrac{dy}{dx}\right]_{x=-\frac{1}{3}}$

8. A body travels so that its distance s at any time t is given by the forumula $s = 3.2t^2$ $(t \geqslant 0)$. Use the delta method to find its speed at instant t.

9. Find the derived functions of the following functions:

   (a) $f(x) = x$ (b) $h(x) = \dfrac{13}{2}$

18.6 Key Concepts

Average speed	Real functions
Instantaneous speed	difference quotients
Limits	Derivatives

18.7 Suggestions for Further Reading

An elegant discussion of the problem of speed is given in

[1] Morris Kline **Calculus—An Intuitive and Physical Approach, Part One** John Wiley and Sons, Inc, New York (U S A.). 1967

For a general reading about the problems studied in Calculus, the reader is referred to the paperback

[2] W.W Sawyer **What is Calculus About ?**
The L W Singer Company, New York (U S A). 1961

For a rigorous treatment of limits of sequences and functions, the reader is referred to

[3] R Courant and H Robbins **What is Mathematics ?**
Oxford University Press, New York (U S A). 1963

An excellent collection of articles on history, pedagogy and concepts of Calculus is found in

[4] T M Apostol, etc (Editors) **Selected Papers on Calculus**
Mathematical Association of America. (U S.A). 1969

UNIT XIX

DERIVATIVES OF POLYNOMIALS

We learn how to find the derivative of natural number powers of x, using the delta method. Then we study derivatives of polynomials

19.1 Introduction

We already know how to find the derivatives of simple functions such as x^2, ax^2 and ax^2+3. We will now learn how to find derivative of a natural number power of x. However, before we do so, we will use the delta method and find the derivative of a constant and derivative of $f(x)=x^3$

19.2 Derivative of a Constant

Let $\quad f(x)=c \quad$ be a constant function.

Then, $\quad f(x+\Delta x)=c \quad$ (Why ?)

[Because, in $f(x)=c$, the value of the function is c, regardless of the value of x.]

Thus, $\quad f(x+\Delta x)-f(x)=c-c=0$

And, $$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{0}{\Delta x}=0 \qquad (1)$$

Now, what is the limit in (1) as $\Delta x \to 0$. Certainly an expression which is always zero must have 0 as the limit (as $\Delta x \to 0$). Thus

$$f'(x)=\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=0$$

In other words, **the derivative of a constant function is zero.**

[The reader is also referred to Q. No. 9(*b*), Exercise 18.2.]

We will now find the derivative of $f(x)=x^3$ in the next section.

19.3 Derivative of $f(x)=x^3$

We first state, without proof, **two important properties of limits** which we shall often use.

If a and c are real numbers and if $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist, then

(*i*) $\lim\limits_{x\to c}\left[af(x)\right]=a\lim\limits_{x\to c}\left[f(x)\right]$

(*ii*) $\lim\limits_{x\to c}\left[f(x)+g(x)\right]=\lim\limits_{x\to c} f(x)+\lim\limits_{x\to c} g(x)$

We now use the delta method to find the derivative of $f(x)=x^3$.

We have $f(x+\Delta x)=(x+\Delta x)^3$

i.e., $f(x+\Delta x)=x^3+3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3$

Thus, $f(x+\Delta x)-f(x)=3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3$ (1)

Dividing both sides of (1) by Δx, we get

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=3x^2+3x(\Delta x)+(\Delta x)^2 \qquad (2)$$

In (2), we need to take the limit as $\Delta x\to 0$. Since $\Delta x\to 0$, it is reasonable to consider only those values of Δx for which $-1<\Delta x<1$

i.e., $|\Delta x|<1$ (3)

Let us multiply both sides of (3) by $|\Delta x|$. We get

$$|\Delta x|^2<|\Delta x|$$

Thus if $\Delta x\to 0$, certainly $(\Delta x)^2\to 0$

Hence, $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\left[3x^2+3x(\Delta x)+(\Delta x)^2\right]$

$=3x^2$

i.e., $f'(x)=3x^2$

Thus, the **derivative of $f(x)=x^3$ is $f'(x)=3x^2$.**

[The reader is advised to verify that the derivative of $h(x)=ax^3$, where 'a' is a real constant is $h'(x)=a(3x^2)=3ax^2$. In other words, **the derivative of a scalar multiple of x^3 is the scalar multiple of the derivative of x^3.**]

19.4 Table of Derivatives

By now, we should be immediately able to write down the derivatives of x^2, ax^2, x^3, ax^3, ax and constants. For the benefit of the reader, we summarize some of these important functions and their derivatives in the table below*.

Function $f(x)$	*Derived Function* $f'(x)$
c (constant)	0
ax	a
x^2	$2x$
ax^2	$a(2x)=2ax$
x^3	$3x^2$
ax^3	$a(3x^2)=3ax^2$

We consider some examples.

Example 1: If $y=-3x^3$, use the table to find $\frac{dy}{dx}$ and hence its value at $x=-2$.

Solution: We at once write down the derivative of $y=-3x^3$.

We have, $\frac{dy}{dx}=-3(3x^2)=-9x^2$

Thus, $\frac{dy}{dx}$ at $(x=-2)$ = Value of $-9x^2$ at $(x=-2)$

$$=-36.$$

Example 2 : Given $f(x)=\frac{7}{4}x^2$, find $f'\left(\frac{1}{7}\right)$.

Solution : We must first find $f'(x)$ and then its value at $\left(x=\frac{1}{7}\right)$

Since $f'(x)=\frac{7}{4}x^2$, we have

$$f'(x)=\frac{7}{4}(2x)=\frac{7}{2}x$$

Thus, $f'\left(\frac{1}{7}\right)$ = Value of $f'(x)$ at $\left(x=\frac{1}{7}\right)$

$$=\frac{7}{2}\left(\frac{1}{7}\right)=\frac{1}{2}$$

*'a', in the table, is intended to be a (real) constant.

In the next section, we learn how to find the derivative of x^n for n, a natural number

19.5 Derivative of $f(x)=x^n$ for n, a natural number

Let us first see if you can guess what the derivative will be. We recall that the

Derivative of x is 1,

Derivative of x^2 is $2x$,

Derivative of x^3 is $3x^2$.

What, do you think, will be the derivative of x^n ? A little use of intuition tells us that it will be nx^{n-1}. That it is indeed so is shown below

$$f(x)=x^n \tag{1}$$

Then $$f(x+\triangle x)=(x+\triangle x)^n$$

We can use the Binomial Theorem to expand $(x+\triangle x)^n$. We have

$$f(x+\triangle x) =x^n+C(n, 1)\, x^{n-1}\, (\triangle x) +C(n, 2)x^{n-2}(\triangle x)^2+ \ldots +C(n,\ n-1)x(\triangle x)^{n-1}+(\triangle x)^n \tag{2}$$

Subtracting (1) from (2) and dividing both sides by $\triangle x$, we have

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}=C(n,\ 1)x^{n-1}+C(n,\ 2)\ x^{n-2}(\triangle x)+\ldots+C(n,\ n-1)x(\triangle x)^{n-2}+(\triangle x)^{n-1} \tag{3}$$

In (3), we need to take the limit as $\triangle x\to0$. Since $\triangle x\to0$, again we will consider only those values of $\triangle x$ for which $-1<\triangle x<1$, i e.

$$|\triangle x| <1 \tag{4}$$

Let us multiply both sides of (4) by $|\triangle x|$ We get

$$|\triangle x|^2<|\triangle x| \tag{5}$$

Thus, if $\triangle x\to0$, certainly $(\triangle x)^2\to0$

Now let us multiply both sides of (4) by $|\triangle x|^2$. We get

$$|\triangle x|^3 < |\triangle x|^2 \tag{6}$$

But, from (5), $|\triangle x|^2 < |\triangle x|$ Thus, (6) becomes

$$|\triangle x|^3 < |\triangle x|$$

Again if $\triangle x\to0$, certainly $(\triangle x)^3\to0$

Similarly, we can show that if $\triangle x \to0$, then $(\triangle x)^4\to0$, $(\triangle x)^5\to0$, and so on.

When we take the limit in (3), therefore, as $\triangle x \to 0$, we get

$$\lim_{\triangle x \to 0} \frac{f(x+\triangle x)-f(x)}{\triangle x} = \lim_{\triangle x \to 0}\Big[C(n,1)\,x^{n-1}+C(n,2)x^{n-2}(\triangle x)+\ldots$$

$$+C(n,n-1)x(\triangle x)^{n-2}+(\triangle x)^{n-1}\Big]$$

i.e., $\qquad f'(x)=C(n,1)x^{n-1}=nx^{n-1}$

We, thus, have the following very important result which we state as a theorem

Theorem 1 : If $f(x)=x^n$,

then the derivative of $f(x)$ is

$$f'(x)=nx^{n-1}$$

where n is a natural number.

[The reader is advised to verify that the derivative of $g(x)=ax^n$ is $g'(x)=a(nx^{n-1})=nax^{n-1}$, where '$a$' is a real constant and n is a natural number. In other words, **the derivative of a scalar multiple of x^n is the scalar multiple of the derivative of x^n for n, a natural number.**]

We now consider some examples

Example 1: If $y=x^6$, find $\frac{dy}{dx}$

Solution. As a direct consequence of the above theorem, we immediately have

$$\frac{dy}{dx}=6x^5$$

Example 2 Given $f(x)=-2x^4$, find $f'(x)$. Hence, find $f'(0)$

Solution : We are given that

$$f(x)=-2x^4$$

Thus, $\qquad f'(x)=-2(4x^3)=-8x^3$

Hence, $\qquad f'(0)=\text{Value of } f'(x) \text{ at } (x=0)$

$$=-8x^3\big]_{x=0}$$

i.e., $\qquad f'(0)=0$

Example 3 : Given $f(x)=\frac{3}{4}x^6$ find $f'\left(\frac{1}{2}\right)$

Solution: We must first find $f'(x)$ and then the value of $f'(x)$ at $x=\frac{1}{2}$
We have

$$f'(x)=\frac{3}{4}\,(6x^5)=\frac{9}{2}x^5$$

Thus, $$f'\left(\frac{1}{2}\right)=f'(x)\Big]_{x=\frac{1}{2}}=\frac{9}{2}\,x^5\Big]_{x=\frac{1}{2}}$$

$$=\frac{9}{2}\left(\frac{1}{2}\right)^5=\frac{9}{64}$$

EXERCISE 19.1

Find the derivative of each of the following functions .

1. $y=-\frac{1}{2}x$
2. $f(x)=7x$
3. $y=3x^2$
4. $y=-\frac{2}{5}x^2$
5. $y=-6x^3$
6. $f(x)=8x^4$
7. $y=\frac{5}{2}x^7$
8. $y=-\frac{3}{8}x^8$
9. $y=-3$
10. $f(x)=7$

For each of the following functions, evaluate the derivative at the indicated value (s) .

11. $s=16\,t,\ t=-3,\ t=0,\ t=18$
12. $s=4.9\,t^2;\ t=1,\ t=5$

13. $y=\frac{3}{10}x^{10}$, $x=-\frac{1}{3}$, $x=0$, $x=2$

14. $f(x)=x^5$, $x=0$, $x=1$, $x=2$, $x=3$

15. $g(x)=4x^9$, $x=-\frac{1}{2}$, $x=\frac{1}{2}$

16. $f(x)=-x$, $x=1802$

17. Given $V=\frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$ and hence $\left.\frac{dV}{dr}\right]_{r=2}$

18. Use the delta method to find the derivative of $f(x)=x^4$. Hence, find $f'\left(-\frac{1}{2}\right)$ and $f'(0)$.

19.6 Derivatives of Polynomials

We now learn how to find the derivatives of polynomials. Consider, for instance, the polynomial

$$f(x)=3x^3-2x \tag{1}$$

Let us use the delta method. We have

$$f(x+\triangle x)=3(x+\triangle x)^3-2(x+\triangle x)$$

i.e.,
$$f(x+\triangle x)=3x^3+9x^2(\triangle x)+9x(\triangle x)^2+3(\triangle x)^3-2x-2(\triangle x) \tag{2}$$

Subtracting (1) from (2) and dividing both sides by $\triangle x$, we get

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}=9x^2+9x(\triangle x)+3(\triangle x)^2-2 \tag{3}$$

Now as $\triangle x\to 0$, by an argument similar to the one given in Section 19.5, we get

$$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=9x^2-2$$

i.e.,
$$f'(x)=9x^2-2$$

We, therefore, see that if

$$f(x)=3x^3-2x$$

then,
$$f'(x)=9x^2-2$$

But the derivative of $3x^3$ by itself, is $9x^2$ and that of $2x$, by itself, is 2. We observe, therefore, **that the derivative of the difference of two (monomials) functions is the difference of their derivatives.**

We now prove following important theorems about derivatives.

Theorem 2: If

$$\mathbf{f(x)=ag(x),}$$

then $$\mathbf{f'(x)=ag'(x)}$$

where 'a' is a real constant.

Proof : The proof is rather simple We use the delta method. We have

$$f(x)=ag(x) \tag{1}$$

Thus, $$f(x+\triangle x)=ag(x+\triangle x) \tag{2}$$

Subtracting (1) from (2) and dividing both sides by $\triangle x$, we get

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}=\frac{ag(x+\triangle x)-ag(x)}{\triangle x}$$

i.e., $$\frac{f(x+\triangle x)-f(x)}{\triangle x}=a\left[\frac{g(x+\triangle x)-g(x)}{\triangle x}\right] \tag{3}$$

In (3), we take the limit as $\triangle x \to 0$. We get

$$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=\lim_{\triangle x\to 0} a\left[\frac{g(x+\triangle x)-g(x)}{\triangle x}\right] \tag{4}$$

We recall the property of limits namely, the limit of a scalar multiple of a function is equal to the scalar multiple of the limit of the function (See Section 19.3). We thus have from (4).

$$f'(x)=a\left[\lim_{\triangle x\to 0}\frac{g(x+\triangle x)-g(x)}{\triangle x}\right]=ag'(x),$$

which proves the theorem

Thus, the derivative of a scalar multiple of a **function is the scalar multiple of the derivative of the function.**

Theorem 3 . If $$\mathbf{f(x)=g(x)+h(x),}$$

then $$\mathbf{f'(x)=g'(x)+h'(x)}$$

That is to say. the **derivative of the sum of two functions is the sum of their derivatives.**

Proof : Again, the proof is rather simple and uses the property of the limit of the sum of two functions We have

$$f(x)=g(x)+h(x) \tag{1}$$

Thus, $$f(x+\triangle x)=g(x+\triangle x)+h(x+\triangle x) \tag{2}$$

And, $$\frac{f(x+\triangle x)-f(x)}{\triangle x}=\frac{g(x+\triangle x)+h(x+\triangle x)-g(x)-h(x)}{\triangle x}$$

$$=\frac{g(x+\triangle x)-g(x)}{\triangle x}+\frac{h(x+\triangle x)-h(x)}{\triangle x} \tag{3}$$

Taking limit in (3) as $\triangle x \to 0$, we get

$$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}=\lim_{\triangle x\to 0}\left[\frac{g(x+\triangle x)-g(x)}{\triangle x}+\frac{h(x+\triangle x)-h(x)}{\triangle x}\right]$$

$$=\lim_{\triangle x\to 0}\frac{g(x+\triangle x)-g(x)}{\triangle x}+\lim_{\triangle x\to 0}\frac{h(x+\triangle x)-h(x)}{\triangle x}.$$

i.e., $$f'(x)=g'(x)+h'(x),$$

which proves the theorem.

As a consequence of Theorems 2 and 3, we have the following important theorem about the derivative of a linear sum of functions, which we simply state but which we shall not prove

Theorem 4 : The derivative of a linear sum of a finite number of functions is the linear sum of their derivatives. That is, if

$$\mathbf{f(x)=a_1f_1(x)+a_2f_2(x)+\ldots+a_nf_n(x)}$$

then,

$$\mathbf{f'(x)=a_1\, f_1'(x)+a_2\, f_2'(x)+\ldots+a_n\, f_n'(x)}$$

where $a_1, a_2, \ldots, a_n$ are (real) constants.

We are now in a position to find the derivatives of polynomials. We consider some examples.

Example 1 : Find $\frac{dy}{dx}$ if $y=x^4-3x^2+x$

Solution : We can use the theorem about the derivative of a linear sum We have

$$y=x^4-3x^2+x$$

Thus, $$\frac{dy}{dx}=4x^3-3(2x)+1$$

$$=4x^3-6x+1$$

Example 2 : Let $f(x)=3x^3+7x^5$. Find $f'(2)$

Solution : We first find $f'(x)$ and then find the value of $f'(x)$ at $x=2$. We have

$$f(x)=3x^3+7x^5$$

Thus,
$$f'(x)=3(3x^2)+7(5x^4)$$
$$=9x^2+35x^4$$
$$f'(2)=\text{Value of } f'(x) \text{ at } (x=2)$$
$$=9(2)^2+35(2)^4$$
$$=36+560=596$$

Example 3 : The height above the ground of a ball thrown upwards with an initial speed of 3m/sec is

$$s=30t-4.9t^2$$

Find its instantaneous speed at the end of 2 seconds.

Solution : To find the instantaneous speed, we must first find the derived function $\dfrac{ds}{dt}$ We have

$$s=30t-4.9t^2$$

Thus,
$$\frac{ds}{dt}=30-4.9(2t)$$
$$=30-9.8t$$

Now, instantaneous speed at the end of 2 seconds

$$=30-9.8t\Big]_{t=2}$$
$$=30-9.8(2)$$
$$=10.4$$

Thus, the instantaneous speed at the end of 2 seconds is 10.4 m/sec.

We close this section by stating the following theorem about the derivatives of polynomials. The proof of the theorem is rather obvious.

Theorem 5 : If

$$\mathbf{f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n}$$

is a polynomial in x, of degree n ($a_n \neq 0$) where a_0, a_1, a_2, ..., a_n are (real) constants, then

$$\mathbf{f'(x)=a_1+2a_2x+3a_3x^2+\ldots+na_nx^{n-1}}$$

We observe that the **derivative is again a polynomial in x, but of degree (n−1).**

EXERCISE 19.2

1. Use the delta method to prove that if

$$y = ax^2 + bx + c,$$

then

$$\frac{dy}{dx} = 2ax + b$$

where a, b and c are (real) constants.

Find the derivative of each of the following:

2. $y = \frac{x^3}{3} - 2x^2 + 6x - 3$

3. $y = \frac{ax + b}{c}$, $c \neq 0$

4. $u = \frac{t^5}{5} - \frac{t^3}{3} + t$

5. $y = x^3 - 6x^2 + 8x$

6. $y = 3x^6 - \frac{x^4}{4}$

7. $y = 4x^2 - 8x + \frac{3}{7}$

8. $f(x) = x^3 - 3x^2 + 3x - \frac{2}{5}$

9. $G(t) = 3t^4 - 5t^3 + t + 1$

10. $H(u) = \frac{u^8}{8} - \frac{u^6}{6} + \frac{u^4}{4} - 2$

11. $G(y) = -2y^{10} + 1 - y^8 + 7y^5$

12. $F(t) = \frac{t^4}{4} - \frac{t^2}{2} + t$

13. $H(y) = \frac{y^3 - 3y^2 + 3y - 6}{3}$

14. Given $u=7t^4-2t^3-8t-5$, find $\frac{du}{dt}$. Hence find $\frac{du}{dt}$ at $t=0$, 1 and 2

15. Given $f(x)=\frac{x^3}{3}-\frac{x^2}{2}+x-16$, find $f'(0)$, $f'(1)$ and $f'(-1)$

16. Given $G(u)=2-u+\frac{u^5}{5}$, find $G'(-2)$

17. Let $G(x)=7x^{10}+5x^3-3$. Find $G(1)$ and $G'(-1)$

18. Let $H(y)=-2y^4-6y^3+2y-4$. Find $H'(2)$.

19. If $y=-\frac{x^4}{4}+\frac{3}{7}x^7-5+2x$, find $\frac{dy}{dx}$ at $x=-2$.

20. Given $f(t)=\frac{2}{9}t^4-\frac{5}{3}t^3+2t-$ [illegible], find $f'(1)$, $f'(3)$ and $f'(-3)$

21. A ball is thrown vertically upwards with an initial speed of 29.4 m/sec. The height s above the ground at any instant t is given by

$$s=29.4t-4.9t^2$$

(*a*) Find the instantaneous speed at the end of 1 second ; 2 seconds

(*b*) How much time does it take for the ball to reach its highest point ?

[**Hint :** At the highest point, the instantaneous speed of the ball becomes zero. The ball then changes its direction of motion and starts falling to the ground]

(*c*) How high will the ball go ?

22. A particle is moving along a horizontal line. Its distance s (in metres) from a point O at t seconds is given by the equation

$$s=8-t^2+t^3$$

Determine its instantaneous speed at the end of 3 seconds.

23. The total cost $C(x)$, in rupees, of manufacturing x watches in a certain factory is given by the **cost function**

$$C(x)=60+400x+40x^2$$

Find the marginal cost of manufacturing 100 watches.

[**Hint :** If the total cost is denoted by $C(x)$, marginal cost is defined as $C'(x)$.]

19.7 Key Concepts

Derivative of a constant	Derivative of a sum of functions
Derivative of x^n	Derivative of a linear sum of functions
Derivative of a scalar multiple of a function	Derivative of a polynomial

19.8 Suggestions for Further Reading

A discussion, at a more rigorous level, is found in

[1] G.H Hardy : A Course of Pure Mathematics 10th Edition. Cambridge University Press, London (U.K.) 1963

UNIT XX

THE PROBLEM OF TANGENTS

In this unit, we attempt to find a geometrical meaning of the derivative Applications to determine the equations of the tangent and the normal at any point of the path of a moving body are considered

20.1 Introduction

We remarked in Section 18 2 that, related to the **problem of speed** of a moving body is the problem of finding the direction of motion of the body at any point of its path. This is essentially the problem of finding the direction of tangent to the path, which we term as the **problem of tangents.**

If the path of a moving body is a circle C with centre O, we recall, from our study of Geometry, that given a point P on C, any line through P meets the curve C in two points, say, P and Q, most of the time. (See Fig. 20.1). This line PQ is called a **secant** to the circle C. Also, if we turn the line PQ around P in such a way that Q comes closer to P, there is a position of the line in which it meets the circle only at one point, namely, at P. When a line meets a circle in only one point, it is called a **tangent** or **tangent line** to the circle. The point at which the line meets the circle is called the **point of contact** and the line is said to **touch** the circle at that point.

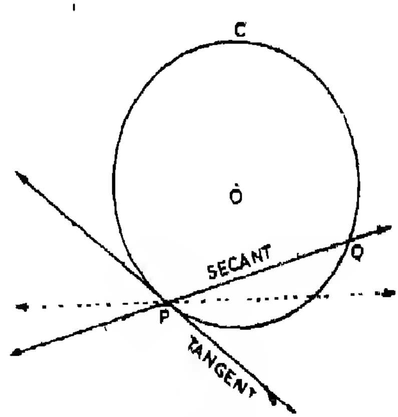


Fig 20.1

We also know how to draw this tangent. We make use of the important theorem, namely, **a tangent at any point of a circle is perpendicular to the radius through the point of contact**

Now if the path of a moving body is other than a circle, (for instance, a parabola, an ellipse, etc) we do not even know what we mean by a tangent at a point P of the path. Can we simply say that a l ne which meets the path in only one point is a tangent to the path at that point ? A moment's reflection will show that this definition will not suffice. Consider, for instance, the following situation .

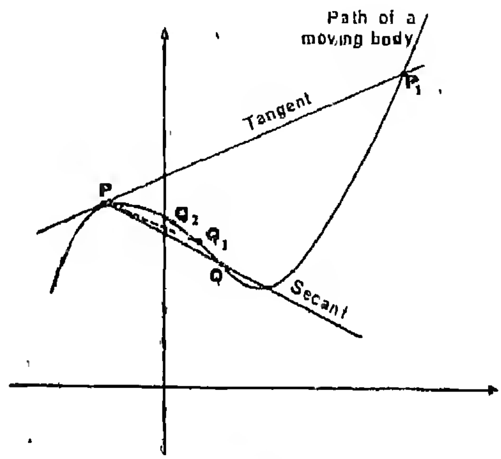


Fig. 20.2

In Fig 20.2, seca t PQ is turned around P in such a way that the point Q comes closer to P. We feel uitively that the position PP_1, therefore should be called a tangent' to the path a. k But PP_1 meets the curve in more than one point. We need a new, more general, defin tion of **tangent** or **tangent line**, namely **it is the limiting line of the secants PQ as Q move towards P along the curve.** The point P is called the **point of tangency.**

Now what do we me n by t e **direction** of the tangent ? We recall from our study of Analytic Geometry h t he direction of a line is determined by its slope. **Thus, when we speak of the direc on of the tangent to the path, we mean to study the slope of the tangent to the path.**

20.2 Geometrical Mean ng of the I erivative

We now attempt to find a geometrical meaning of the derivative Let us assume that the path of a moving body is a parabola given by the equation

$$y \quad \frac{1}{4} x^2+2$$

We recall the various steps in finding the derivative at a point, say (x_0, y_0) to the above path and see what is the goemetrical meaning of each step [See Figs. 20.3 (*i*) nd (*ii*)]

We substitute (x_0, y_0) in the equation and get

$$y_0=\frac{1}{4}x_0^2+2 \qquad \text{Geometrically} \quad \frac{1}{4}x_0^2+2=PA$$

Now $y_0+\Delta y=\frac{1}{4}\left(x_0+\Delta x\right)^2+2 \qquad \frac{1}{4}(x_0+\Delta x)^2+2=QB$

Thus, $\Delta y=\frac{1}{4}\left(x_0+\Delta x\right)^2+2-\left(\frac{1}{4}x_0^2+2\right) \qquad \Delta y=QB-PA$

i e., $\Delta y=\frac{1}{2}x_0\left(\Delta x\right)+\frac{1}{4}\left(\Delta x\right)^2 \qquad$ i.e., $\Delta y=QB-MB-QM$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x}=\frac{1}{2}x_0+\frac{1}{4}\left(\Delta x\right) \qquad \frac{\Delta y}{\Delta x}=\frac{QM}{PM}=\text{slope of the secant } PQ$$

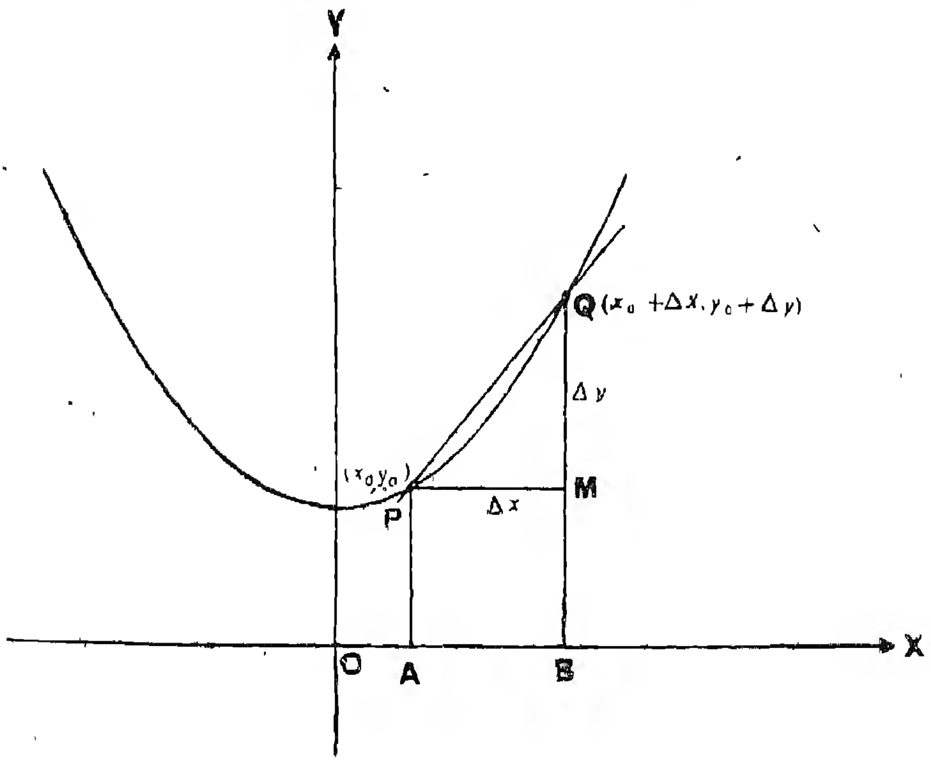


Fig 20 3 (*i*)

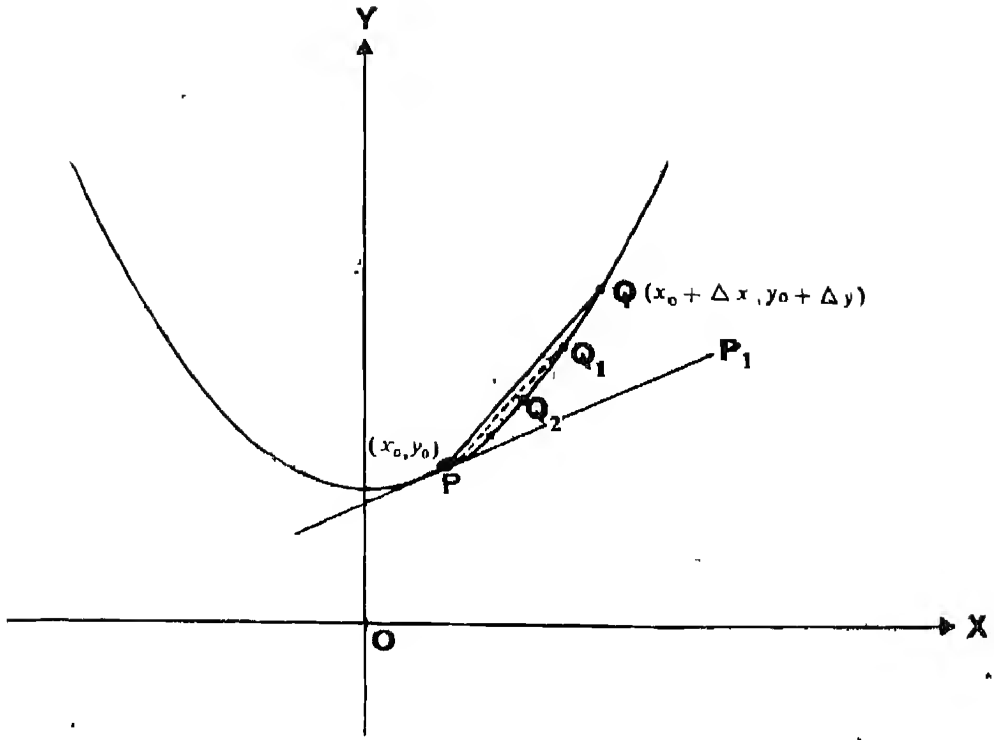


Fig 20 3 (ii)

The quantity $\frac{\Delta y}{\Delta x}$ has an important geometrical meaning, namely, it is **the slope of the secant PQ**.

Now, let us take the limit as $\Delta x \to 0$ We get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{1}{2}x_0 + \frac{1}{4}(\Delta x)\right]$$

i.e.,
$$\frac{dy}{dx} = \tfrac{1}{2}x_0$$

Corresponding to this last step of taking the limit as $\Delta x \to 0$, let us see what is happening geometrically. As Δx approaches zero (*i,e.*, as $|\Delta x|$ becomes smaller and smaller), the point Q moves along the curve towards P—*i.e.*, Q moves through positions Q_1, Q_2, etc [See Fig. 20.3 (*ii*)]. We observe that as $|\Delta x|$ becomes smaller and smaller, $|\Delta y|$ simultaneously becomes smaller and smaller. Also that as $\Delta x \to 0$, the lines PQ, PQ_1, PQ_2, etc. approach the line PP_1 which, of course, is the tangent to the curve at P. Thus, as $\Delta x \to 0$, the slope of the secant PQ approaches the slope of the tangent at P.

We see, therefore, that **the derivative at a point P of the curve is the slope of the tangent to the curve at the point P.**

Let us now denote the path of a moving body by

$$y=f(x)$$

where $f(x)$ has derivative $f'(x)$ at all points of the path

Let (x_0, y_0) be a point on this path.

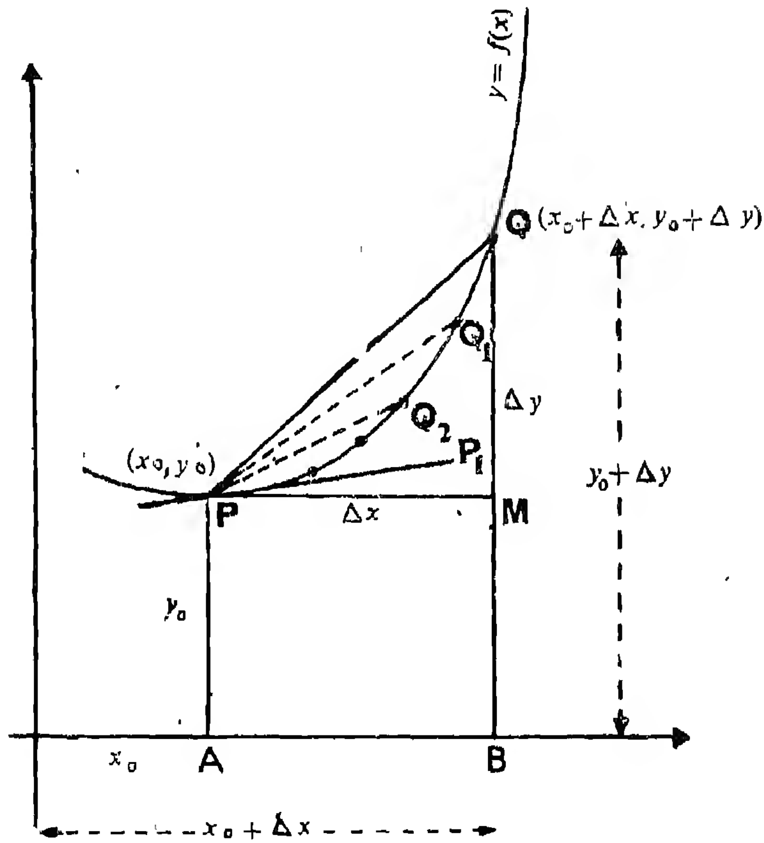


Fig. 20.4 (i)

Then the slope of the secant PQ is

$$m_{PQ}=\frac{f(x_0+\Delta x)-f(x_0)}{(x_0+\Delta x)-x_0}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \tag{1}$$

Now as $\Delta x \to 0$, R.H.S of (1) and, hence, m_{PQ} approaches $f'(x_0)$. Also as $\Delta x \to 0$, the point Q moves along the curve towards* P and the secant PQ moves towards

*In other words, $f(x_0+\Delta x)$ approaches $f(x_0)$ as $\Delta x \to 0$. In technical language, this fact is expressed by saying that the function $f(x)$ is *continuous* at x_0.

the tangent line PP_1 at P [See Figs. 20.4(i) and 20.4(ii)]. Consequently m_{PQ} approaches the slope m of the tangent line PP_1. Thus $m=f'(x_0)$.

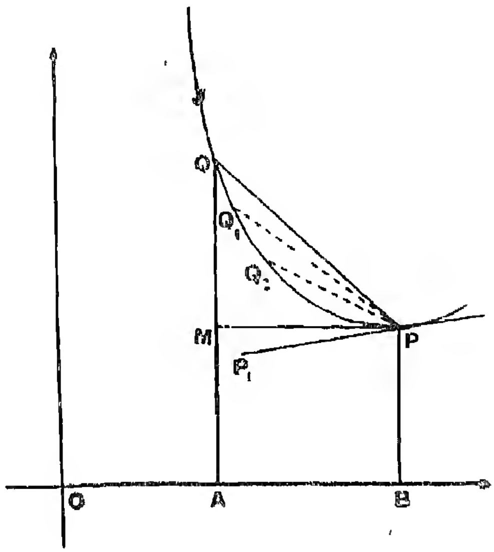


Fig 20.4 (ii)

Hence the derivative at the point P of a curve is the slope of the tangent to the curve at the point P.

This is the geometrical meaning of the derivative at a point. It provides us with a method of finding the tangent to a curve at any point on the curve.

It also provides us with a method of finding the direction of a moving body at any point of its (curvilinear) path. **We define the slope of the path or curve at any point to be** the slope of the tangent at that point, which can be easily determined since it is the derivative at that point.

EXERCISE 20.1

1. Find the slope of the tangent to each of the following curves

   (i) $s=4.9t^2$ at $t=1$.

   (ii) $y=2x^2$ at $x=-1$

   (iii) $y=16-x^2$ at $x=0$

   (iv) $h=-16t^2+128t$ at $t=$

2. Find the slope of the tangent to the curve

$$y=x^3$$

at $x=-1$, $x=0$ and $x=1$

20.3 Applications of the Derivative : Tangents and Normals

We have already seen that if a curve is given by the equation $y=f(x)$ where $f(x)$ has a derivative $f'(x)$ at every point, then the derivativ at a point P of the curve is the slope of the tangent to the curve at the point P. Thus, it is simple to **find the equation of the tangent at any point of the curve by using the point-slope form.** The tangents in all these cases have finite slopes and thus are not parallel to y axis

However, we come across curves which have tangents parallel to y-axis at some point(s) [See Fig 20 5]

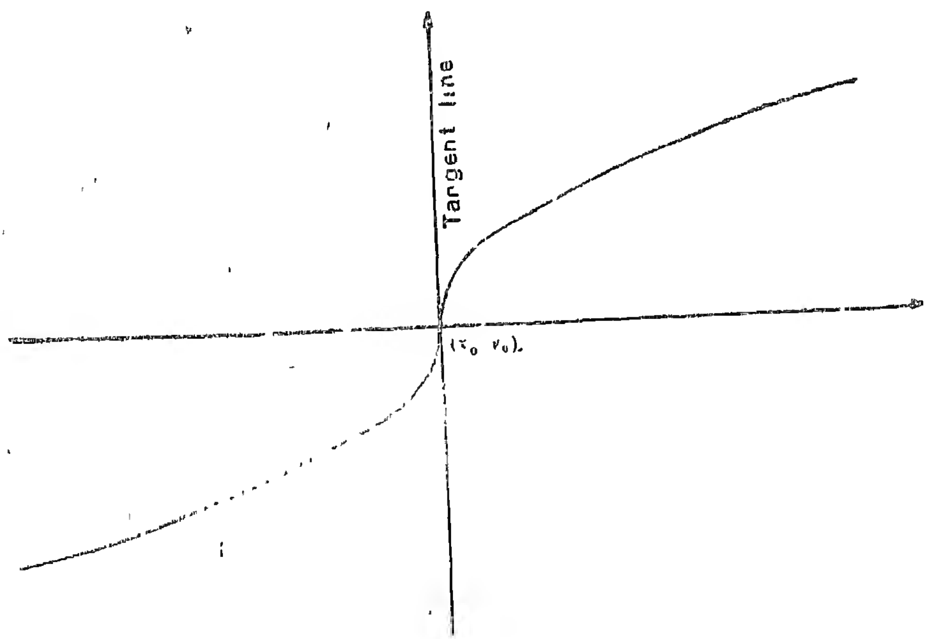


Fig. 20 5

At such points, $\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ does not approach a finite limit as $\Delta x \to 0$.

The tangent line, in such cases, has the equa ion $x=x_0$.

We now define the normal to a curve. We say that **the normal line to a curve at a given point is the line perpendicular to the tangent line at that point.** [See Fig. 20.6]. We recall from Analytic Geometry that **the slope of the normal line at a given point is the negative of the reciprocal of the slope of tangent line at that point.** Thus, again we can use the point-slope form and determine the equation of the normal line at a given point.

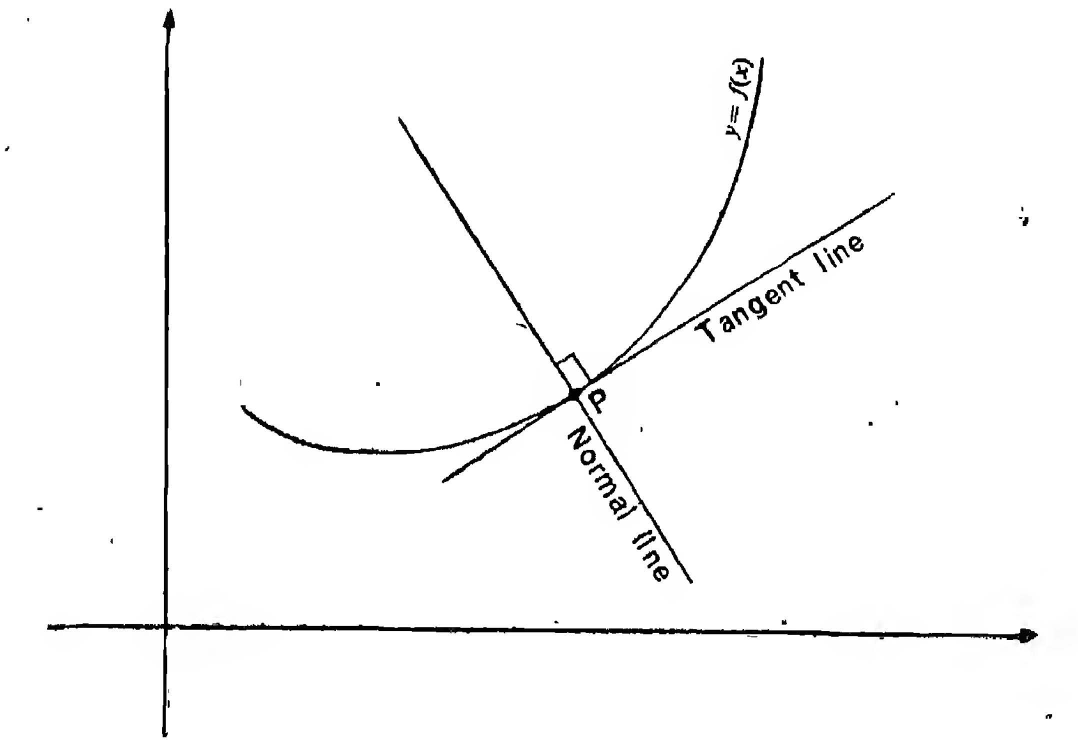


Fig 20 6

We consider some examples.

Example 1 : Find the equation of the tangent to

$$y=x^2-4x+2$$

at the point (4 2)

Solution · We are given that

$$y=x^2-4x+2$$

We have, $$\frac{dy}{dx}=2x-4$$

And. $$\left.\frac{dy}{dx}\right]_{x=4}=\left.2x-4\right]_{x=4}=4$$

Thus the slope of the tangent at (4, 2) is 4 Using the point-slope form, we have the equation of the tangent as

$$y-2=4(x-4)$$

Or, $$y-4x+14=0$$

Hence, $y-4x+14=0$ is the equation of the tangent to the given curve at the point (4, 2)

Example 2 : Find the equation of the tangent line to

$$y=2x^2+7$$

which is parallel to the line $4x-y+3=0$

Solution : We are given that

$$y=2x^2+7$$

We have $$\frac{dy}{dx}=4x$$

Thus the slope of the tangent at a point (x,y) to $y=2x^2+7$ is $4x$

If this tangent is to be parallel to

$$4x-y+3=0,$$

then the slope of the tangent must equal the slope of the line, which is 4

Thus, $4x = 4$

Whence, $x = 1$

And, $y = 2x^2+7\Big]_{(x=1)} =9$

Hence, at the point (1,9) on the curve

$$y = 2x^2+7,$$

the tangent line will be parallel to the given line.

It is now easy to find the equation of this tangent We have

$$y-9=4(x-1)$$

Or, $$y-4x-5=0$$

Thus, $y-4x-5=0$ is the equation of the required tangent line

Example 3 : Determine the points on the curve

$$y=x^3-3x^2-9x+7$$

at which the tangents are parallel to the x-axis (i.e., the tangents are horizontal).

Solution : We are given that

$$y=x^3-3x^2-9x+7$$

We have $$\frac{dy}{dx} = 3x^2-6x-9$$

When a tangent is parallel to the x-axis, we recall that its slope is zero. Thus

$$3x^2-6x-9=0$$

Or, $$x^2-2x-3=0$$

Or, $$(x+1)(x-3)=0$$

Whence, $$x=-1, 3$$

Now, when $x=-1$, $y=(-1)^3-3(-1)^2-9(-1)+7=12$

And, when $x=3$, $y=-20$

Thus, $(-1,12)$ and $(3,-20)$ are the required points at which the tangents to the curve

$$y=x^3-3x^2-9x+7$$

are parallel to the x-axis

Example 4 : Find the equation of the normal to

$$y=\frac{1}{4}x^3$$

at the point $(4,16)$

Solution : We are given that

$$y=\frac{1}{4}x^3$$

We have $$\frac{dy}{dx}=\frac{3}{4}x^2$$

And $$\left.\frac{dy}{dx}\right]_{x=4}=\left.\frac{3}{4}x^2\right]_{x=4}=12$$

Thus, the slope of the tangent at (4,16) is 12. What will, therefore, be the slope of the normal at (4,16)? We recall, it will be the negative reciprocal of the slope of the tangent at the given point. Thus, the slope of the normal at (4,16) will be $-\frac{1}{12}$.

The equation of the normal at (4,16) is, therefore,

$$y-16=-\frac{1}{12}(x-4)$$

Or, $$12y+x-196=0$$

Hence, $x+12y-196=0$ is the required equation of the normal

Example 5 : Find an equation of the normal line to

$$y=x^3+2x+6$$

which is parallel to the line

$$14y+x+4=0$$

Solution : We are given that

$$y=x^3+2x+6$$

We have $$\frac{dy}{dx}=3x^2+2$$

Thus, the slope of the tangent at a point (x,y) to

$$y=x^3+2x+6$$

is $3x^2+2$ and hence, that of the normal at the point (x,y) is $-\frac{1}{3x^2+2}$.

If this normal is to be parallel to $14y+x+4=0$, we have

$$-\frac{1}{3x^2+2}=-\frac{1}{14} \qquad \text{(Why ?)}$$

Whence, $$3x^2+2=14$$

Or, $$x^2=4$$

Hence, $$x=2,\ -2$$

When $x=2$, $y=(2)^3+2(2)+6=18$

When $x=-2$, $y=(-2)^3+2(-2)+6=-6$

Thus we have two normals, one at (2, 18) and the other at (−2, −6) which are parallel to the given line. Let us find their equations. We have

$$y-18=-\frac{1}{14}(x-2)$$

Or, $14y+x-254=0$

And, $y-(-6)=-\frac{1}{14}(x+2)$

Or, $14y+x+86=0$

Hence, $14y+x-254=0$ and $14y+x+86=0$ are the required normals.

EXERCISE 20.2

Find the equation of the tangent to each of the following curves at the indicated point :

1. $y=x^2+2$ at (2, 6)
2. $y=16-x^2$ at (0, 16)
3. $y=-16x^2+128x$ at (4,256)
4. $y=x^2-4x-5$ at (−2, 7)
5. $y=x^3$ at (−3, −27)
6. Find the point(s) on the curve $y=x^2-4x+2$ where the slope of the tangent is 10.
7. Find the point(s) on the curve $y=\frac{1}{4}x^3$ where the slope of the tangent is $\frac{16}{3}$.
8. Determine the point(s) on the curve $y=x^2+1$ at which the slope of the tangent is equal to the
   (a) x-coordinate,
   (b) y-coordinate.
9. Determine the point(s) on the curve $y=x^3$ at which the slope of the tangent is equal to the
   (a) x—coordinate,
   (b) y—coordinate
10. Prove that no two tangent lines to the curve $y=x^2$ are parallel.

11. Prove that the x-axis is a tangent to the curve $y=x^3$ at the point (0, 0).

12. Find the point of intersection of the tangents to the curve $y=2x^2$ at the points (1, 2) and (−1, 2).

13. Prove that the tangents to $y=x^3$ at $x=1$ and $x=-1$ are parallel.

14. Find the equation of the tangent line to the curve $y=x^2+4x-16$ which is parallel to the line $6x-2y-3=0$

15. Find the point on the curve $y=2x^2-3x+5$ at which the tangent makes an angle of 45° with the positive direction of the x-axis.

[**Hint :** Slope of the tangent=tan 45°]

Find the equation of the normal to each of the following curves at the indicated point :

16. $y=x^2+2x+1$ at (2, 9)

17. $y=\frac{1}{4}x^3$ at (2, 2)

18. $y=-x^3+2x$ at (−1, −1)

19. $y=5-6x-x^2$ at (0, 5)

20 $y=2x^3-x^2+3$ at (1, 4)

21 Find equations of the normal lines to the curve $y=x^3-3x$ which are parallel to the line $9y+x=4$.

22. Prove that the equation of the normal to $x^2=4ay$ at the point $(2a, a)$ is $x+y=3a$

23. Find the point on the curve $y=2x^2-6x-4$ at which the tangent is parallel to the x-axis.

24. Determine the point of the curve $y=3x^2-5$ at which the tangent is perpendicular to a line whose slope is $-\frac{1}{3}$.

25 Find the equation of the normal to the curve

$$x^2+2y=8x-2$$

at the point (6, 5).

20.4 Key Concepts

Secant	Derivative as a slope
Tangent or tangent line	Equation of the tangent or tangent line
Point of contact or tangency	Equation of the normal or normal line
Slope of the tangent	

20.5 Suggestions for Further Reading

An excellent discussion of the problem of tangents is given in [1] and [2], Section 18 7.

The reader is also referred to

[1] Louis Leithold : **The Calculus Book : A First Course with Applications and Theory.**
Harper and Row, Publishers, Inc. New York (U.S.A.). 1971.

UNIT XXI

DERIVATIVES OF PRODUCTS AND QUOTIENTS OF FUNCTIONS

We learn how to find the derivatives of the product and quotient of two functions. The **Quotient Rule** *is applied to find the derivatives of negative integral powers of x. We then study the method of finding the derivative of a composite function by the* **Chain Rule** *Finally, derivatives of implicit functions are studied and their application to determine the derivative of rational number powers of x are considered*

21.1 Introduction

We now learn the method of finding the derivative of the product and of the quotient of two functions. But first some terminology.

Given a function $f(x)$, the process of calculating its derivative $f'(x)$ is called **differentiation.** Also, another name for the derivative of $f(x)$ is the **differential coefficient** of $f(x)$.

We will need two **important properties of limits,** which we state, without proof. Let $f(x)$ and $g(x)$ be given functions. If c is a real number and if $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist, then

(i) $\lim\limits_{x\to c}\left[f(x).g(x)\right]=\left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right]$

(ii) $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$ provided $\lim\limits_{x\to c} g(x)\neq 0$.

[**Note :** For property (ii), the domain of g may be R or some suitable subset of R]

21.2 Derivative of the Product of Two Functions

Let $f(x)$ be the product of two functions $g(x)$ and $h(x)$, *i.e.*,

$$f(x)=g(x)h(x)$$

We wish to find $f'(x)$. We will use the delta method. We have

$$f(x+\Delta x)=g(x+\Delta x)\ h(x+\Delta x)$$

Thus, $\quad f(x+\Delta x)-f(x)=g(x+\Delta x)\ h(x+\Delta x)-g(x)\ h(x)$

Dividing both sides by Δx, we get

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{g(x+\Delta x)\ h(x+\Delta x)-g(x)\ h(x)}{\Delta x} \tag{1}$$

We want to find $f'(x)$. Therefore, in (1), we need to take the limit, as $\Delta x \to 0$. Let us consider the expression in the R.H.S. of (1).

A moment's reflection will show that we must add and subtract certain term(s) to the numerator to express it as a combination of terms whose limits we already know. Let us add and subtract the term $g(x+\Delta x)\ h(x)$. [We can, just as well, add or subtract the term $g(x)\ h(x+\Delta x)$.] We get

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{g(x+\Delta x)\ h(x+\Delta x)-g(x+\Delta x)\ h(x)+g(x+\Delta x)\ h(x)-g(x)\ h(x)}{\Delta x}$$

i.e., $$\frac{f(x+\Delta x)-f(x)}{\Delta x}=g(x+\Delta x)\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]+h(x)\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] \tag{2}$$

Taking the limit as $\Delta x \to 0$, we get

$$f'(x)=\lim_{\Delta x \to 0}\left\{g(x+\Delta x)\left[\frac{h(x+\Delta x)-h(x)}{\Delta x}\right]+h(x)\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\right\}$$

$$=\lim_{\Delta x \to 0}\left[g(x+\Delta x)\left(\frac{h(x+\Delta x)-h(x)}{\Delta x}\right)\right]+\lim_{\Delta x \to 0}\left[h(x)\left(\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)\right] \qquad \textbf{(Why?)}$$

$$=\lim_{\Delta x \to 0} g(x+\Delta x)\lim_{\Delta x \to 0}\frac{h(x+\Delta x)-h(x)}{\Delta x}+h(x)\lim_{\Delta x \to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} \qquad \textbf{(Why ?)}$$

Now how do we find $\lim_{\Delta x \to 0} g(x + \Delta x)$? Let us consider

$$\lim_{\Delta x \to 0} \left[g(x + \Delta x) - g(x) \right]$$

We have $\lim_{\Delta x \to 0} \left[g(x + \Delta x) - g(x) \right]$

$$= \lim_{\Delta x \to 0} \left[\frac{g(x + \Delta x) - g(x)}{\Delta x} . \Delta x \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \lim_{\Delta x \to 0} \left[\Delta x \right] \qquad \text{(Why ?)}$$

$$= g'(x).0 = 0$$

i.e., $\lim_{\Delta x \to 0} \left[g(x + \Delta x) - g(x) \right] = 0$

Thus, $\lim_{\Delta x \to 0} g(x + \Delta x) = \lim_{\Delta x \to 0} g(x) = g(x)$

Hence,

$$f'(x) = g(x)\, h'(x) + h(x)\, g'(x)$$

We, therefore, see that **the derivative of the product of two functions is first function times the derivative of the second plus second function times the derivative of the first.** We call this the **Product Rule** of differentiation

We state this important rule as a Theorem

Theorem 6 : (Product Rule) If

$$f(x) = g(x)\, h(x),$$

then $f'(x) = g(x)\, h'(x) + h(x)\, g'(x)$

It is often convenient to denote the derivative of a function $f(x)$ by $D_x[f(x)]$ or $D[f(x)]$ Using this notation, we write the **Product Rule** as

$$f'(x) = g(x)\ D[h(x)] + h(x)\ D[g(x)]$$

We consider some examples.

Example 1. Find the derivative of

$$f(x) = (x^3 - 3x^2 + 4)(4x^5 + x^2 - 1)$$

Solution We will use the **Product Rule.** We have

$$f'(x) = (x^3 - 3x^2 + 4)D(4x^5 + x^2 - 1) + (4x^5 + x^2 - 1)D(x^3 - 3x^2 + 4)$$

Thus, $f'(x) = (x^3 - 3x^2 + 4)(20x^4 + 2x) + (4x^5 + x^2 - 1)(3x^2 - 6x)$

$$= (20x^7 - 60x^6 + 82x^4 - 6x^3 + 8x)$$
$$+ (12x^7 - 24x^6 + 3x^4 - 6x^3 - 3x^2 + 6x)$$

i.e., $f'(x) = 32x^7 - 84x^6 + 85x^4 - 12x^3 - 3x^2 + 14x$

Example 2 Differentiate

$$f(x)=(3x^2+2)^2$$

Solution : We write $f(x)$ as product of two functions and use the **Product Rule.** We have

$$f(x)=(3x^2+2)(3x^2+2)$$

Thus, $f'(x)=(3x^2+2)D(3x^2+2)+(3x^2+2)D(3x^2+2)$

$=(3x^2+2)(6x)+(3x^2+2)(6x)$

$=12x(3x^2+2)$

Hence, $f'(x)=36x^3+24x$

[**Note** : Later we shall learn an important theorem to be able to differentiate such functions without writing them as products and using the **Product Rule.**]

Example 3: It is easy to extend **the Product Rule to the product of more than two functions.** For instance, if $f(x)$ is the product of three functions $u(x)$, $v(x)$ and $w(x)$, i e.,

If $\mathbf{f(x)=u(x)v(x)w(x)}$

then, $\mathbf{f'(x)=u(x)v(x)w'(x)+u(x)v'(x)w(x)+u'(x)v(x)w(x)}$

Using the D-notation for the derivative, we write

$$\mathbf{f'(x)=u(x)v(x)D[w(x)]+u(x)w(x)D[v(x)]+v(x)w(x)D[u(x)]}$$

Find the derivative of

$$f(x)=(x^2+2)(x^3-3x^2+4)(x^4-1)$$

Solution : We are given

$$f(x)=(x^2+2)(x^3-3x^2+4)(x^4-1)$$

Thus, $f'(x)=(x^2+2)(x^3-3x^2+4)D(x^4-1)$

$+(x^2+2)(x^4-1)\ D(x^3-3x^2+4)$

$+(x^3-3x^2+4)(x^4-1)D(x^2+2)$

$=(x^2+2)(x^3-3x^2+4)(4x^3)+(x^2+2)(x^4-1)(3x^2-6x)$

$+(x^3-3x^2+4)(x^4-1)(2x)$

$=(4x^8-12x^7+8x^6-8x^5+32x^3)+(3x^8-6x^7+6x^6-12x^5-3x^4$

$+6x^3-6x^2+12x)+(2x^8-6x^7+8x^5-2x^4+6x^3-8x)$

i e., $f'(x)=9x^8-24x^7+14x^6-12x^5-5x^4+44x^3-6x^2+4x$

EXERCISE 21.1

1. Prove that if

$$f(x)=u(x)v(x)w(x),$$

then $$f'(x)=u(x)v(x)w'(x)+u(x)v'(x)w(x)+u'(x)v(x)w(x)$$

Find the derivative of each of the following:

2. $f(x)=(x+1)(2x-9)$

3. $g(x)=\left(\frac{3}{2}x+7\right)(2x^2+1)$

4. $F(x)=x^2(2x^2+3x+8)$

5. $G(x)=(\sqrt{2}x^3+x^5)(\sqrt{3}x^2+\frac{1}{5}x^5)$

6. $f(x)=(ax^2-b)(cx^2-d)$

7. $f(x)=(x^2+2)^2$

8. $y=(x+1)^3$

9. $g(x)=x(x-3)(x^2+x)$

10. $g(x)=\left(\frac{3}{2}x^2+1\right)^2(x+2)$

•11. $G(x)=(x^4-1)(5x^3+6x)$

21.3 Derivative of the Quotient of Two Functions

Before we find the derivative of the quotient of two functions, we will learn how to find the derivative of the reciprocal of a function. We prove the following important theorem:

Theorem 7 : If $h(x)\neq 0$ and if $f(x)=\frac{1}{h(x)}$,

then, $$f'(x)=-\frac{h'(x)}{[h(x)]^2}$$

Proof : We are given that

$$f(x)=\frac{1}{h(x)}$$

Thus, $$f(x+\Delta x)=\frac{1}{h(x+\Delta x)}$$

And, $$f(x+\Delta x)-f(x)=\frac{1}{h(x+\Delta x)}-\frac{1}{h(x)}$$

i.e., $$f(x+\Delta x)-f(x)=\frac{h(x)-h(x+\Delta x)}{h(x+\Delta x)h(x)} \qquad (1)$$

Dividing both sides of (1) by $\triangle x$, we get

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}=\frac{h(x)-h(x+\triangle x)}{h(x+\triangle x)h(x)(\triangle x)} \tag{2}$$

Again we want to find $f'(x)$. We, therefore, write (2) as

$$\frac{f(x+\triangle x)-f(x)}{\triangle x}=-\frac{1}{h(x)}\left[\frac{h(x+\triangle x)-h(x)}{\triangle x}\right]\left[\frac{1}{h(x+\triangle x)}\right] \tag{3}$$

Taking limit in (3) as $\triangle x\to 0$, we get

$$f'(x)=-\frac{1}{h(x)}\lim_{\triangle x\to 0}\left[\frac{h(x+\triangle x)-h(x)}{\triangle x}\right]\left[\frac{1}{h(x+\triangle x)}\right]$$

$$=-\frac{1}{h(x)}\lim_{\triangle x\to 0}\left[\frac{\dfrac{h(x+\triangle x)-h(x)}{\triangle x}}{h(x+\triangle x)}\right]$$

i.e.,
$$f'(x)=-\frac{1}{h(x)}\left[\frac{\lim\limits_{\triangle x\to 0}\dfrac{h(x+\triangle x)-h(x)}{\triangle x}}{\lim\limits_{\triangle x\to 0}h(x+\triangle x)}\right]$$

But we have already seen in the proof of Theorem 6, that

$$\lim_{\triangle x\to 0} h(x+\triangle x)=h(x)$$

Thus $f'(x)=-\dfrac{1}{h(x)}\dfrac{h'(x)}{h(x)}$

i.e.,
$$f'(x)=-\frac{h'(x)}{[h(x)]^2}$$

which proves the theorem.

We can now find the derivative of quotient of two functions. We have the following important theorem:

Theorem 8 : If $h(x)\neq 0$ and if $f(x)=\dfrac{g(x)}{h(x)}$

then
$$f'(x)=\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}$$

Proof : We write

$$f(x)=g(x)\left[\frac{1}{h(x)}\right]$$

and use the Product Rule and Theorem 7 to find $f'(x)$. We have

$$f'(x)=g(x)D\left[\frac{1}{h(x)}\right]+\frac{1}{h(x)}\,D[g(x)]$$

$$=g(x)\left(\frac{-h'(x)}{[h(x)]^2}\right)+\frac{1}{h(x)}\,g'(x)$$

$$=\frac{-g(x)h'(x)+h(x)g'(x)}{[h(x)]^2}$$

i.e.,

$$f'(x)=\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}$$

which proves the theorem.

Using the D-notation, we write

$$f'(x)=\frac{h(x)D[(g(x)]-g(x)D[h(x)]}{[h(x)]^2}$$

Or,

$$f'(x)=\frac{\text{Denom}\times D\ [\text{Num}]-\text{Num}\times D\ [\text{Denom}]}{[\text{Denom}]^2}$$

where, in the second expression, Denom and Num are abbreviations for denominator and and numerator respectively. We call this the **Quotient Rule** of differentiation.

We now consider some examples.

Example 1 : Differentiate

$$f(x)=\frac{1}{(3x-1)}$$

Solution : We are given that

$$f(x)=\frac{1}{(3x-1)}$$

Using Theorem 7, we have

$$f'(x)=-\frac{D[(3x-1)]}{[(3x-1)]^2}=-\frac{3}{(3x-1)^2}$$

Example 2 : Find the derivative of

$$f(x)=\frac{x+1}{x^2+1}$$

Solution : We use the **Quotient Rule** to find the derivative. We have

$$f'(x)=\frac{(x^2+1)D[x+1]-(x+1)\,D[x^2+1]}{(x^2+1)^2}$$

$$=\frac{(x^2+1)(1)-(x+1)(2x)}{(x^2+1)^2}$$

i.e.,

$$f'(x)=\frac{-x^2-2x+1}{(x^2+1)^2}$$

21.4 Derivative of f $(x)=x^{-n}$ $(x\neq 0)$ when $-n$ is a negative integer

Since $x^{-n}=\dfrac{1}{x^n}$, we can use Theorem 7 or the **Quotient Rule** to find the derivative of x^{-n}. We have

$$f(x)=\frac{1}{x^n}$$

Thus,

$$f'(x)=\frac{-(nx^{n-1})}{[x^n]^2}$$

$$=\frac{-nx^{n-1}}{x^{2n}}$$

i.e.,

$$f'(x)=\frac{-n}{x^{n+1}}=-nx^{-n-1}$$

We thus have the following theorem.

Theorem 9 : If $-n$ is a negative integer and if f $(x)=x^{-n}$ $(x\neq 0)$,
then $\quad$ **$f'(x)=-nx^{-n-1}$ $(x\neq 0)$**

In the examples and exercises that follow, we leave it to the reader to specify the domains of functions.

We consider some examples.

Example 1 : Find the derivative of

$$f(x)=\frac{2}{x^5}$$

Solution : We have

$$f(x)=\frac{2}{x^5}=2x^{-5}$$

Thus,

$$f'(x)=2(-5x^{-5-1})$$

$$=-10x^{-6}$$

i.e.,

$$f'(x)=\frac{-10}{x^6}$$

Example 2 : Differentiate

$$y=x^3+\frac{2}{x^2}-\frac{1}{x}$$

Solution : We have

$$y=x^3+\frac{2}{x^2}-\frac{1}{x}$$

Thus, $$\frac{dy}{dx}=D\left(x^3+\frac{2}{x^2}-\frac{1}{x}\right)$$ (Why ?)

$$=D(x^3)+D(2x^{-2})-D(x^{-1})$$

$$=3x^2-4x^{-3}+x^{-2}$$

Hence, $$\frac{dy}{dx}=3x^2-\frac{4}{x^3}+\frac{1}{x^2}$$

EXERCISE 21.2

Find the derivative of each of the following .

1. $f(x)=\dfrac{1}{x+1}$
2. $g(x)=\dfrac{(x+2)^2}{x-1}$
3. $f(x)=x^{-7}$
4. $y=\dfrac{1}{\sqrt{3x^3}}$
5. $y=\dfrac{x^2+5x}{2x^3}$
6. $g(x)=\dfrac{x}{2x-1}$
7. $h(x)=\dfrac{1}{x^4}-\dfrac{3}{x-1}+5x^{-2}+10$
8. $y=x^4-3+x^{-1}-4x^{-4}$
9. $g(x)=\dfrac{2x^2-1}{2x+3}$
10. $f(x)=\dfrac{x^6+3x^5-2x^2+7}{x^4}$
11. $g(x)=\dfrac{x^2-1}{x^2+1}$
12. $f(x)=\dfrac{(x+2)(3x-1)}{2x+5}$

13. $g(x)=\dfrac{x^3-8}{(x+2)^2}$

14. $H(x)=\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

15. $y=\dfrac{x(x^2+3)}{x-2}$

16. $f(x)=\dfrac{2x^3+x^2+5}{x-4}$

17 $g(x)=\dfrac{1}{(x-a)(x-b)(x-c)}$

21.5 The Derivative of a Composite Function : The Chain Rule

So far we have learnt how to differentiate when, say, y is given as a function of x. Often, we come across situations when y is a function of, say, u and u, in turn, is a function of, say, x For instance, we have the situation where

$$y=f(u)=2u^3 \tag{1}$$

and, $$u=g(x)=3x^3-4x^2+7 \tag{2}$$

The equations (1) and (2) together define y as a function of x.

For, if we replace u in (1) by the right-hand side of (2), we get

$$y=h(x)=f[g(x)]=2[(3x^3-4x^2+7)^3]$$

We say y is a **function of a function** or that y is a **composite function**

[The reader has already encountered a composite function in Example 2, Section 21 2. We differentiated it by using the Product Rule]

We now state and prove a theorem for finding the derivative of a composite function.

Theorem 10 : If y is a function of u, defined by y=f(u), and if u is a function of x, defined by u=g(x), then y is a function of x and

$$\frac{dy}{dx}=\frac{dy}{du}\quad\frac{du}{dx}$$

Proof : Since $y=f(u)$ and $u=g(x)$, we have
$$y=f[g(x)]$$

We wish to find $\dfrac{dy}{dx}$ We use the delta method We have

$$y+\Delta y=f[g(x+\Delta x)]$$

Thus, $$\Delta y=f[g(x+\Delta x)]-f[g(x)]$$

Dividing both sides by Δx, we have

$$\frac{\Delta y}{\Delta x}=\frac{f[g(x+\Delta x)]-f[g(x)]}{\Delta x} \quad (1)$$

In (1), taking the limit as $\Delta x \to 0$, we get

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left[\frac{f[g(x+\Delta x)]-f[g(x)]}{\Delta x}\right] \quad (2)$$

How shall we find the limit in the R.H.S. of (2)? We observe that since

$$u=g(x),$$

$$u+\Delta u=g(x+\Delta x) \quad (3)$$

Substituting (3) in (2), we get

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left[\frac{f(u+\Delta u)-f(u)}{\Delta x}\right] \quad (4)$$

Now we write* the expression in the brackets in R H S. of (4) as

$$\frac{f(u+\Delta u)-f(u)}{\Delta u}\cdot\frac{\Delta u}{\Delta x} \quad \text{(Why ?)} \quad \text{We get}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left[\frac{f(u+\Delta u)-f(u)}{\Delta u}\quad\frac{\Delta u}{\Delta x}\right].$$

i.e., $$\frac{dy}{dx}=\left[\lim_{\Delta x\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u}\right]\left[\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right] \quad \text{(Why ?)} \quad (5)$$

Now, from (3) as $\Delta x\to 0$, $\Delta u\to 0$. Thus (5) can be rewritten as

$$\frac{dy}{dx}=\left[\lim_{\Delta u\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u}\right]\left[\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] \quad (6)$$

i.e., $$\frac{dy}{dx}=f'(u)g'(x)$$

Since $$y=f(u),$$

$$\frac{dy}{du}=f'(u)$$

Also since $$u=g(x),$$

---

*We tacitly assume that $\Delta u\neq 0$ for sufficiently small Δx, since most of the functions that we come across in this book will have this property

$$\frac{du}{dx}=g'(x)$$

Thus (6) can be rewritten as

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

which proves the theorem

We often refer to Theorem 10 as the Chain Rule.

We now apply **Chain Rule** to the example with which we started Section 21.5. We have

$$y=2u^3 \tag{7}$$

and $$u=3x^3-4x^2+7 \tag{8}$$

Thus, by the **Chain Rule**

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Now, from (7),

$$\frac{dy}{du}=6u^2$$

And, from (8),

$$\frac{du}{dx}=9x^2-8x$$

Thus, $$\frac{dy}{dx}=6u^2(9x^2-8x)$$

i.e., $$\frac{dy}{dx}=6(3x^3-4x^2+7)^2(9x^2-8x)$$ (Why ?)

We consider some examples

Example-1 : Given $y=(3x^2+2)^2$, find $\frac{dy}{dx}$.

[See also Example 2, Section 21.2]

Solution : We let $y=u^2$ where $u=3x^2+2$.

Then, $$\frac{dy}{du}=2u \quad \text{and} \quad \frac{du}{dx}=6x$$

Thus by the **Chain Rule**, we get

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=2u(6x)=12ux$$

i.e., $$\frac{dy}{dx}=12x(3x^2+2)$$

Example 2 : Find $\frac{dy}{dx}$ if $y=\left(4x^2+\frac{5}{2}x+9\right)^{10}$

Solution : We will use the **Chain Rule** We let

$$y=u^{10} \text{ where } u=4x^2+\frac{5}{2}x+9$$

Then, $$\frac{dy}{du}=10u^9 \text{ and } \frac{du}{dx}=8x+\frac{5}{2}$$

Hence, $$\frac{dy}{dx}=10u^9\left(8x+\frac{5}{2}\right)=10\left(4x^2+\frac{5}{2}x+9\right)^9\left(8x+\frac{5}{2}\right)$$

Example 3 : Given $f(x)=(x^2+1)^3(3x^2-5x)^2$, find $f'(x)$.

Solution : Clearly $f(x)$ is the product of two functions $g(x)=(x^2+1)^3$ and $h(x)=(3x^2-5x)^2$. Thus, we use the **Product Rule** and have

$$f'(x)=(x^2+1)^3\, D[(3x^2-5x)^2]+(3x^2-5x)^2\, D[(x^2+1)^3]$$

Now we can use the **Chain Rule and get**

$$f'(x)=(x^2+1)^3[2(3x^2-5x)(6x-5)]+(3x^2-5x)^2[3(x^2+1)^2(2x)]$$

Thus, $$f'(x)=2(x^2+1)^3(3x^2-5x)(6x-5)+6x(x^2+1)^2(3x^2-5x)^2$$

$$=2(x^2+1)^2(3x^2-5x)\,[(6x-5)(x^2+1)+3x(3x^2-5x)]$$

i.e., $$f'(x)=2(x^2+1)^2(3x^2-5x)(15x^3-20x^2+6x-5)$$

Example 4 : If $y=\left(\frac{3x-1}{2x+1}\right)^3$, find $\frac{dy}{dx}$

Solution : We let $y=u^3$ where $u=\frac{3x-1}{2x+1}$

Then, $\frac{dy}{du}=3u^2$ and $\frac{du}{dx}=\frac{(2x+1)3-(3x-1)2}{(2x+1)^2}=\frac{5}{(2x+1)^2}$

Thus, $\frac{dy}{dx}=3u^2.\ \frac{5}{(2x+1)^2}$

i.e., $\frac{dy}{dx}=15\,\frac{(3x-1)^2}{(2x+1)^4}$

EXERCISE 21.3

Use the **Chain Rule** to find the derivative of each of the following functions :

1. $f(x)=(x+1)^6$

2. $g(x)=(2-3x)^3$

3. $h(x)=(x^3+2)^4$

4. $f(x)=(x^2+2x-11)^5$

5. $h(x)=(x-1)^{-3}$

6. $g(x)=(2x^2+5x-3)^{-4}$

7. $f(x)=(3x^2+2)^3(5x-1)^2$

8. $g(x)=(x^2+3)^4(x^2+5)^2$

9. $f(t)=\left(\dfrac{2t^3+1}{3t^2+1}\right)^2$

10. $g(x)=(2x-3)^{-1}(4x+3)^{-2}$

11. $f(x)=(x^3-2x)^4(x^2+7)^{-2}$

12. $g(t)=\dfrac{3}{(2t^2+5)^2}$

13. $f(x)=\dfrac{3}{2-x}$

14. $y=\left(\dfrac{5x}{3x+2}\right)^3-2\left(\dfrac{5x}{3x+2}\right)^2$

15. $f(x)=(x+1)(x+2)(x+3)$

Find $\dfrac{dy}{dx}$ in each of the following

16. $y=\dfrac{1}{4}v^4,\quad v=\dfrac{2}{3}x^3+5$

17. $y=\dfrac{3-u}{2+u},\quad u=\dfrac{4x}{1-x^2}$

18. $y=9v^2,\quad v=1-\dfrac{3}{2}x^2$

19. If $y=(ax+b)^m(cx+d)^n$ where m and n are integers, find $\dfrac{dy}{dx}$

20. A particle is moving with a velocity, $v=\frac{1}{4}(t^2+5)^2$. Determine its acceleration at any instant t

[**Hint**: Acceleration $a=\frac{dv}{dt}$]

21. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+3)$. Determine the rate of change of its volume with respect to x.

[**Hint**. Volume $V=\frac{4}{3}\pi r^3$]

21.6 Derivative of Implicit Functions

We have learnt how to differentiate functions, such as $y=x^3-5x+6$, $y=(x^2-2)^4$, etc. We observe that, in the functions which we have encountered so far, y is **defined explicitly in terms of x**, i.e., we can write

$$y=f(x) \text{ where } f(x)=x^3-5x+6, \text{ etc}$$

But not all functions are defined explicitly. Consider, for instance, the equation

$$2y^3+x^3+xy^2-y=5 \tag{1}$$

We observe that it is not possible in (1) to solve explicitly for y. But there may exist one or more functions $f(x)$ such that (1) is satisfied when $y=f(x)$. **In such cases we say y is defined implicitly in terms of x.**

[The reader may recall the equation of a circle $x^2+y^2=r^2$ or that of a parabola $y^2=4ax$. In both these equations, y is defined **implicitly** in terms of x. We knew that there are two functions $y=\sqrt{r^2-x^2}$ and $y=-\sqrt{r^2-x^2}$ each of which satisfy the equation of the circle. Similarly, there are two functions $y=\sqrt{4ax}$ and $y=-\sqrt{4ax}$, each of which satisfy the equation of the parabola. We thus observe that $x^2+y^2=r^2$ represents both the functions $y=\sqrt{r^2-x^2}$ and $y=-\sqrt{r^2-x^2}$ implicity. Similar statement can be made with regard to the equation $y^2=4ax$.]

Let us see **how we can find the derivative of y with respect to x when y is defined implicitly in terms of x.**

We consider (1). We have

$$2y^3+x^3+xy^2-y=5$$

We use the theorems on the derivative of the sum, Product Rule, derivative of x^n when n is natural number and the Chain Rule and (implicitly) differentiate both sides with respect to x. We get

$$2\left(3y^2\frac{dy}{dx}\right)+3x^2+\left[x\left(2y\frac{dy}{dx}\right)+y^2(1)\right]-\frac{dy}{dx}=0$$

i.e
$$6y^2\frac{dy}{dx}+3x^2+2xy\frac{dy}{dx}+y^2-\frac{dy}{dx}=0 \qquad (2)$$

It is now easy to solve (2) for $\frac{dy}{dx}$. We have

$$\frac{dy}{dx}(6y^2+2xy-1)=-(3x^2+y^2)$$

Thus, $\frac{dy}{dx}=-\frac{3x^2+y^2}{6y^2+2xy-1}$ for all values of x and y satisfying (1) and for which $6y^2+2xy\neq 1$.

We consider some examples.

Example 1: Find the derivative of y with respect to x, given the equation
$$-x^4+6x^2y^2+4y^3-2x^2+5y=0$$

Solution. We differentiate both sides of the given equation with respect to x and obtain

$$-4x^3+\left[6x^2\left(2y\frac{dy}{dx}\right)+6y^2(2x)\right]+4\left(3y^2\frac{dy}{dx}\right)-2(2x)+5\left(\frac{dy}{dx}\right)=0$$

Thus, $\frac{dy}{dx}[12x^2y+12y^2+5]=4x^3-12xy^2+4x$

Whence, $\frac{dy}{dx}=\frac{4x^3-12xy^2+4x}{12x^2y+12y^2+5}$ for all values of x and y satisfying the given equation and for which $12x^2y+12y^2+5\neq 0$.

Example 2 : Given the equation

$$(x+y)^2-y^3+(x-y)^2-x^3=0, \text{ find } \frac{dy}{dx}.$$

Solution : We differentiate both sides with respect to x and get
$$2(x+y)\left(1+\frac{dy}{dx}\right)-3y^2\frac{dy}{dx}+2(x-y)\left(1-\frac{dy}{dx}\right)-3x^2=0$$

Thus, $\frac{dy}{dx}[2(x+y)-3y^2-2(x-y)]=3x^2-2(x-y)-2(x+y)$

i.e. $\frac{dy}{dx}(4y-3y^2)=3x^2-4x$

Whence, $\frac{dy}{dx}=-\frac{x(3x-4)}{y(3y-4)}$

for $y\neq 0$ and $y\neq\frac{4}{3}$

Example 3 : Find the equation of the normal line to the curve $2x^3+y^3=10$ at the point (1, 2).

Solution We recall that the slope of the normal line at a point is the negative reciprocal of the slope of the tangent line at the given point How do we find the slope of the tangent line at a point ? We recall that it is the value of the derivative (of y with respect to x) at that point. We, therefore, differentiate both sides of the given equation with respect to x We get

$$6x^2+3y^2\frac{dy}{dx}=0$$

Thus, $$\frac{dy}{dx}=-2\frac{x^2}{y^2} \text{ for } y\neq 0$$

At the point (1, 2), the value of the derivative is

$$-2\frac{x^2}{y^2}\Big]_{(1,\,2)}=-\frac{1}{2}$$

Thus the slope of the tangent to the curve at the point (1, 2) is $-\frac{1}{2}$, whence the slope of the normal to the curve at the point (1, 2) will be +2

We can now use the point-slope form and obtain the equation of the normal line at (1, 2) as

$$y-2=2(x-1)$$

Or, $$y-2x=0$$

We now use the method of implicit differentiation to find the derivative of rational powers of x when $x\neq 0$

21.7 Derivative of $x^{\frac{p}{q}}$ $(x > 0)$ when $\frac{p}{q}$ is a rational number

We let $$y=x^{\frac{p}{q}} \tag{1}$$

Without any loss of generality, we can assume that $q>0$ Raising both sides of (1) to q, we get

$$y^q=x^p \tag{2}$$

We differentiate both sides of (2) with respect to x We get

$$qy^{q-1}\frac{dy}{dx}=px^{p-1}$$

Thus, $$\frac{dy}{dx}=\frac{p}{q}\ \frac{x^{p-1}}{y^{q-1}} \tag{3}$$

But $y=x^{\frac{p}{q}}$ Thus $y^{q-1}=x^{\frac{p(q-1)}{q}}=x^{p-\frac{p}{q}}$

Substituting this value of y^{q-1} in the R.H S. of (3), we have

$$\frac{dy}{dx}=\frac{p}{q}\ \frac{x^{p-1}}{x^{p-\frac{p}{q}}}$$

$$=\frac{p}{q}\ x^{p-1-p+\frac{p}{q}}$$

i.e., $$\frac{dy}{dx}=\frac{p}{q}\ x^{\frac{p}{q}-1}$$

We thus have the following important theorem

Theorem 11 : If $\frac{p}{q}$ is a rational number and if

$$y=x^{\frac{p}{q}} \quad (x>0)$$

then $$\frac{dy}{dx}=\frac{p}{q}\ x^{\frac{p}{q}-1}$$

We consider some examples. In these examples and exercises that follow, we leave it to the reader to specify the domains of the functions

Example 1 : Find the derivatives of each of the following

(*a*) $y=x^{\frac{3}{2}}$ (*b*) $y=\frac{1}{\sqrt{2x}}$

(*c*) $y=x^{\frac{2}{3}}$ (*d*) $y=x^{-\frac{5}{2}}$

Solution : We use Theorem 11 which provides us with the rule for differentiating rational powers of x. We have

(*a*) $$y=x^{\frac{3}{2}}$$

Thus, $$\frac{dy}{dx}=\frac{3}{2}\cdot x^{\frac{3}{2}-1}=\frac{3}{2}x^{\frac{1}{2}}$$

(*b*) $$y=\frac{1}{\sqrt{2x}}=\frac{1}{\sqrt{2}}x^{-\frac{1}{2}}$$

Thus, $$\frac{dy}{dx}=\frac{1}{\sqrt{2}}\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)=-\frac{1}{2\sqrt{2}}x^{-\frac{3}{2}}$$

(*c*) $$y=x^{\frac{2}{3}}$$

Thus, $$\frac{dy}{dx}=\frac{2}{3}x^{\frac{2}{3}-1}=\frac{2}{3}x^{-\frac{1}{3}}.$$

(*d*) $$y=x^{-\frac{5}{2}}$$

Thus, $$\frac{dy}{dx}=\left(-\frac{5}{2}\right)x^{-\frac{5}{2}-1}=-\frac{5}{2}x^{-\frac{7}{2}}$$

Example 2 : Differentiate

$$f(x)=\sqrt[3]{2x^4+x^2-x}$$

with respect to x.

Solution : We have

$$f(x)=(2x^4+x^2-x)^{\frac{1}{3}}.$$

Thus, $$f'(x)=\frac{1}{3}(2x^4+x^2-x)^{\frac{1}{3}-1}D[2x^4+x^2-x]$$

$$=\frac{1}{3}(2x^4+x^2-x)^{-\frac{2}{3}}(8x^3+2x-1)$$

i.e, $$f'(x)=\frac{8x^3+2x-1}{3(2x^4+x^2-x)^{\frac{2}{3}}}$$

Example 3 Find the derivative of

$$f(t)=(t^2+t+5)^{\frac{1}{3}}(t^3+1)^{\frac{2}{3}}$$

with respect to t.

Solution We use the **Product Rule** and have

$$f'(t)=(t^2+t+5)^{\frac{1}{3}}D_t[(t^3+1)^{\frac{2}{3}}]+(t^3+1)^{\frac{2}{3}}D_t[(t^2+t+5)^{\frac{1}{3}}]$$

$$=(t^2+t+5)^{\frac{1}{3}}\left[\frac{2}{3}(t^3+1)^{-\frac{1}{3}}D(t^3+1)\right]$$

$$+(t^3+1)^{\frac{2}{3}}\left[\frac{1}{3}(t^2+t+5)^{-\frac{2}{3}}D(t^2+t+5)\right]$$

$$=(t^2+t+5)^{\frac{1}{3}}\left[\frac{2}{3(t^3+1)^{\frac{1}{3}}}\right]3t^2+(t^3+1)^{\frac{2}{3}}\left[\frac{1}{3(t^2+t+5)^{\frac{2}{3}}}\right](2t+1)$$

$$=\frac{6t^2(t^2+t+5)+(2t+1)(t^3+1)}{3(t^3+1)^{\frac{1}{3}}(t^2+t+5)^{\frac{2}{3}}}$$

i e, $$f'(t)=\frac{8t^4+7t^3+30t^2+2t+1}{3(t^3+1)^{\frac{1}{3}}(t^2+t+5)^{\frac{2}{3}}}$$

Example 4 : Find the derivative of

$$h(x)=\frac{\sqrt{x-1}}{\sqrt{x+1}}$$

Solution : We use the **Quotient Rule** and get

$$h'(x)=\frac{\sqrt{x+1}\,D[\sqrt{x-1}\,]-\sqrt{x-1}\,D[\sqrt{x+1}\,]}{(\sqrt{x+1})^2} \qquad (1)$$

In (1), we need to find $D[\sqrt{x-1}\,]$ and $D[\sqrt{x+1}\,]$. We use the **Chain Rule** and the rule for finding derivatives of rational powers of x and obtain

$$D[\sqrt{x-1}]=D[(x-1)^{\frac{1}{2}}]$$

$$=\frac{1}{2}(x-1)^{-\frac{1}{2}}D(x-1)$$

$$=\frac{1}{2}(x-1)^{-\frac{1}{2}}$$

Similarly, $$D[\sqrt{x+1}]=\frac{1}{2}(x+1)^{-\frac{1}{2}}$$

Substituting the values of these derivatives in (1), we get

$$h'(x)=\frac{\sqrt{x+1}\,\frac{1}{2\sqrt{x-1}}-\sqrt{x-1}\,\frac{1}{2\sqrt{x+1}}}{x+1}$$

$$=\frac{\frac{(x+1)-(x-1)}{2\sqrt{x-1}\,\sqrt{x+1}}}{x+1}=\frac{\frac{1}{\sqrt{x-1}}\,\frac{1}{\sqrt{x+1}}}{x+1}$$

Thus, $$h'(x)=\frac{1}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}$$

Example 5 : Given the equation $x^2+y^2=16$.

(*a*) Express y as two explicit functions of x,

(*b*) Find the derivative of y with respect to x by differentiating the given equation implicitly and verify that the derivative is the same as that obtained by differentiating either of the functions in (*a*)

Solution : (*a*) We are given, $x^2+y^2=16$

Thus, $y^2=16-x^2$

Whence, $y=\pm\sqrt{16-x^2}$

(b) We (implicitly) differentiate both sides of $x^2+y^2=16$ with respect to x. We get

$$2x+2y\frac{dy}{dx}=0$$

Thus, $$\frac{dy}{dx}=-\frac{x}{y}$$

Now consider the function $y=\sqrt{16-x^2}$ We use the Chain Rule and Theorem 11 to find $\frac{dy}{dx}$. We get

$$\frac{dy}{dx}=\frac{1}{2}(16-x^2)^{\frac{-1}{2}}(-2x)$$

i.e., $$\frac{dy}{dx}=\frac{-x}{\sqrt{16-x^2}} \qquad (1)$$

Substituting $\sqrt{16-x^2}=y$ in (1), we get

$$\frac{dy}{dx}=-\frac{x}{y}$$

which is the same derivative that we obtained by implicit differentiation.

[We leave it as an exercise for the reader to show that if $y=-\sqrt{16-x^2}$ then $\frac{dy}{dx}=-\frac{x}{y}$]

EXERCISE 21.4

Find the derivative of y with respect to x in each of the following.

1. $x^3+8xy+y^3=64$
2. $x^6+y^6+6x^2y^2-16=0$
3. $ax^2+2hxy+by^2+2gx+2fy+c=0$
4. $\frac{x^2}{9}+\frac{y^2}{16}=1$
5. $(x-3)^2+(y-5)^2=9$
6. $(2x-3y)^3=3x^3-y^2$
7. Find the equation of the tangent to the curve $20x^4+y^4=36$ at the point (1, 2).

8. Find the equation of the normal line to the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at the point $(-3, 0)$.

9. Determine the equation of normal line to the curve

$$x^2+2y^2-4x-6y+8=0$$

at the points whose abscissa is 2.

Find the derivative of each of the following with respect to x:

10. $y=x^{-\frac{8}{3}}$

11. $y=\frac{4}{3}x^{\frac{3}{4}}$

12. $y=\sqrt{x}-\frac{1}{\sqrt{x}}$

13. $y=(4x^2+5)^{\frac{1}{5}}$

14. $f(x)=\sqrt[3]{ax+b}$

15. $H(x)=(2x^3-5x^2+8)^{-4}(x^4+x-1)^{\frac{1}{2}}$

16. $G(x)=\sqrt[3]{5x-9}\,\sqrt[3]{3x-4}$

17. $y=\frac{\sqrt{1-x}}{\sqrt{x+2}}$

18. $y=\frac{(x^5-2x+1)^{\frac{3}{2}}}{x^2}$

21.8 Key Concepts

Derivative of product of two functions or **Product Rule**	Derivative of a composite function or **Chain Rule**
Derivative of Quotient of two functions or **Quotient Rule**	Derivative of Implicit Functions or **Implicit Differentiation**
Derivative of negative integral powers of x	Derivative of rational number powers of x

21.9 Suggestions for Further Reading

The reader is referred to [2], *Section 20.5 for a further reading, at a slightly more rigorous level, of various theorems on differentiation and their applications.*

A must for every reader is the book by

[1] R. Courant and F. John : **Introduction to Calculus and Analysis, Volume I.** Inter-science Publishers, New York (U S A.). 1965.

MISCELLANEOUS EXERCISE V

(On units XVIII, XIX, XX, XXI)

1. The distance s, at time t, of a particle moving in a straight line is given by the equation

   $s=t^4-18t^2$ Find its speed at $t=10$ seconds.

2. In each of the following cases, find the instantaneous speed of an object moving according to the given law of motion.

   (i) $s=\dfrac{(t^3-1)^2}{t^3}$

   (ii) $s=\sqrt[3]{t^3+10}$

s and t have their usual meanings Find also the speed at $t=3$

3. The number of bacteria in a certain culture, after t hours, is given by the equation

   $$f(t)=0.5t^2+4t+80,\ 0\leqslant t\leqslant 12$$

   Determine the instantaneous rate of change of $f(t)$ at $t=5$

4. The Cost Function $C(x)$, in rupees, of producing x items $(x\geqslant 15)$ in a certain factory is given by

   $$C(x)=20+10x^2+\frac{15}{x}$$

   Determine the Marginal Cost Function and the marginal cost of producing 100 items.

5. The Revenue Function $R(x)$, in rupees, of selling x items $(x\geqslant 0)$, of a certain manufacturing concern is

   $$R(x)=21x-\frac{x^2}{5}$$

   Determine the Marginal Revenue Function and the marginal revenue of selling 25 items.

6. The Profit Function $P(x)$, in rupees, of a certain firm is given by

$$\frac{P(x)}{x}=[1+x(2-x)]$$

Find the Marginal Profit Function.

Find the derived function of each of the following :

7. $f(x)=\frac{1}{7}x^7+\frac{1}{5}x^5-\frac{2}{3}x^3+5$

8. $g(x)=5x^8-6x^3+x^4-10$

9. $f(x)=\frac{2}{5}x^{\frac{2}{3}}-x^{\frac{-4}{5}}+\frac{3}{x^2}$

10. $g(x)=\frac{1}{1-3x}-\sqrt[5]{x^2}$

Find $\frac{dy}{dx}$ for each of the following

11. $y=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(1+x+x^2\right)$

12. $y=(x^2+3x+5)(x^2-2)^2$

13 $y=\left(\frac{x-\sqrt{x}}{1-2x}\right)^2$

14 $y=\left(\frac{1}{1+x}\right)\left(x^{-2}+\frac{2}{x}-1\right)+\sqrt[3]{x}-\frac{1}{\sqrt[3]{x}}$

15 $y=(1-2x^2)^3(1-x)^{\frac{3}{2}}$

16. $y=x+\frac{1}{x-1}+\sqrt{100-x^2}$

17 $y=x\sqrt{9-x^2}+(x+2)\sqrt{4-x}$

Find the derivative of each of the following

18 $S(r)=(3r+4)(\pi r^2+2\pi r)$

19 $V(r)=r(1-r)(\pi r^2+r)$

20. $Q(r)=\frac{\pi^2}{r^4}-r^2$

21. $H(x)=\sqrt[3]{x^2(x^2+3)}$

22. $F(x)=x+\sqrt{x^2+8}$

Use Chain Rule to find $\frac{du}{dx}$ for each of the following :

23. $u=\frac{y}{y-1}$, $y=x-\frac{1}{x}$

24 $u=y^2$, $y=\sqrt{x}+\frac{1}{\sqrt{x}}$

Use Chain Rule to find $\frac{dy}{dx}$ for each of the following

25 $y=at^2$, $t=\frac{x}{2a}$

26. $y=2t+3t^2$, $t=\sqrt{x}$

Find $\frac{dy}{dx}$ in each of the following :

27. $x^2=\frac{x+3y}{x-3y}$

28. $\sqrt{x}+\sqrt{y}=5$

29. $\frac{y}{x+y}=3+x^3$

30. $\frac{1}{x}-\frac{1}{y}-10=0$

31 $x^3+y^3-3axy=0$

Find the slope of the tangent to each of the following curves at the indicated point

32 $y=8x^2-3$ at $x=\frac{1}{4}$

33. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=2$ at $(1,1)$

34. $3x^2+y^3-8=0$ at $(0,2)$

Find the equation of the tangent to each of the following curves at the indicated point.

35. $y=3x^2-5x+2$ at $x=2$

36. $x^2+y^2=25$ at $(4,-3)$

37. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at $(1,1)$

38 $y=\frac{-6}{\sqrt{x}}$ at $(9,-2)$

39 Given the curve $y=\sqrt{4x-3}-1$, find the point at which the tangent has $\frac{2}{3}$ as its slope.

40 Find the point of intersection of the tangents to the curve

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

at the points (4, 0) and (0, 5).

41 Find the equation of the tangent line to the curve

$$y=\sqrt{5x-3}-2$$

which is

(a) parallel to the line $2x-y+9=0$

(b) perpendicular to the line $5y+2\sqrt{2}\,x=13$

42 Determine the equation of the normal line to the curve

$$y=\sqrt[3]{5-x}$$

at the point $(-3, 2)$

43 Find the equation of the normal line to the curve

$$y=\frac{x^2-1}{x^2+1}$$

at $x=1$

44. Find the equation of the normal line to the curve

$$x^2+y^2=16$$

which makes an angle of 30° with the positive direction of x-axis.

45. Determine the equation of the normal line(s) to the curve

$$y=4x^3-3x+5$$

which are parallel to the line $9y+x+3=0$.

UNIT XXII

SIMULTANEOUS LINEAR INEQUATIONS--AN APPLICATION

(An Introduction to Linear Programming)

In this unit, we learn how to use the knowledge of simultaneous linear equations and inequations to solve problems that arise in trade, commerce, industry and military operations However, we consider only the simpler versions of some real life problems from these fields.

22.1 Recall

You should review the following concepts from your earlier classes. In particular, you should recall

Linear inequations in two variables and their solutions

Graph of the solutions or solution-set

Simultaneous linear inequations and their solutions

22.2 Introduction

We will use our knowledge of simultaneous linear equations and inequations to solve problems that arise in trade, commerce, industry, military operations, etc. However, for the sake of simplicity, we will be constrained to consider only the simpler versions of some real life problems from these fields.

Let us consider an example :

Example : A furniture dealer deals in only two items (simplified version of a real-life situation) : tables and chairs. He has Rs 5000.00 to invest and a space to store at most 60 pieces. A table costs him Rs 250.00 and a chair Rs 50.00. He can sell a table at a profit of Rs 50.00 and a chair at a profit of Rs 15.00.

Assuming he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit ?

In this example, we **observe** that there are two types of activities—the dealer can invest his money in purchasing tables or chairs or combinations thereof. Furthermore he would earn different profits by following different investment strategies.

Also, there are certain overriding conditions or constraints, namely, his investment is limited (to a maximum of Rs 5000.00)) and so is his storage space (to a maximum of 60 pieces).

Suppose, for instance, he chooses to buy only tables but no chairs. With his capital of Rs 5000.00, he can buy 20 tables and then his total profit will be Rs 1000.00 (Why?)

If, however, he chooses to buy only chairs but no tables, he can buy 100 chairs with Rs 5000.00. However, he can store only 60 pieces. Therefore, he is forced to buy only 60 chairs which will give him a total profit of Rs 900.00 only. Of course, in this situation he does not invest the entire capital.

There are many other possibilities, for instance, he may choose to buy 10 tables. After buying 10 tables, he will be left with a capital of Rs 2500.00 with which he can buy 50 chairs. This makes 60 pieces which are possible for him to store. In this case, his profit would be Rs$(10 \quad 50+50\times 15)$ *i.e.* Rs 1250.00

The number each of tables and chairs is, therefore, a **variable**. Each is necessarily non-negative (Why ?) The profit of the dealer is a function of both these variables. The dealer, of course, would like to invest in such a way so as to maximize his profit.

The above is a typical problem from among the class of problems, called **optimization problems**, that deal with the allocation of limited resources, under certain overriding conditions or constraints, to obtain the best possible or optimal results in meeting the given objectives.

22.3 Mathematical Formulation of the Problem

Let us read the problem given in Section 22.2 again and formulate it mathematically. We note that

Maximum possible investment		Rs 5000.00
Maximum storage space		60 pieces of furniture
Cost of a		
	table :	Rs 250.00
	chair :	Rs 50.00

Profit on a

table	Rs 50.00
chair	Rs 15.00

Let us denote by x, the number of tables and by y, the number of chairs that the dealer buys Of course,

$$x \geqslant 0 \tag{1}$$

$$y \geqslant 0 \tag{2}$$

The dealer is constrained by the maximum amount he can invest and by the maximum number of pieces he can store Stated mathematically,

$$250x+50y \leqslant 5000$$

or, $$5x+y \leqslant 100 \tag{3}$$

and $$x+y \leqslant 60 \tag{4}$$

The dealer wants to invest in such a way so as to maximize his profit, say, P, which stated as a function of x and y is

$$P=50x+15y \tag{5}$$

Mathematically, therefore, the problem now reduces to

Maximize $P=50x+15y$

subject to the constraints

$$5x+y \leqslant 100$$

$$x+y \leqslant 60$$

$$x \geqslant 0$$

and $$y \geqslant 0$$

The method of maximizing (or minimizing) a linear function of several variables (called OBJECTIVE FUNCTION) subject to the condition that the variables are non-negative and satisfy a set of linear equations and/or inequations (called LINEAR CONSTRAINTS) is given the name LINEAR PROGRAMMING. The term linear implies that all the mathematical relations used in the problem are **linear relations**, while the term **programming** refers to the method of determining a particular **programme** or **plan** of action. The two together have the technical meaning stated above.

22.4 The Graphical* Method of Solving Linear Programming Problems

Let us refer to the problem given in Section 22.3 again and graph the constraints stated as inequations (1), (2), (3) and (4). [See Fig. 22.1] The shaded region consists of

*The analytical methods of solving linear programming problems are beyond the scope of this book.

points in the intersection of all the closed (Why ?) half-planes (Why ?) represented by the four constraints. Each point in this region represents a **feasible choice** open to the

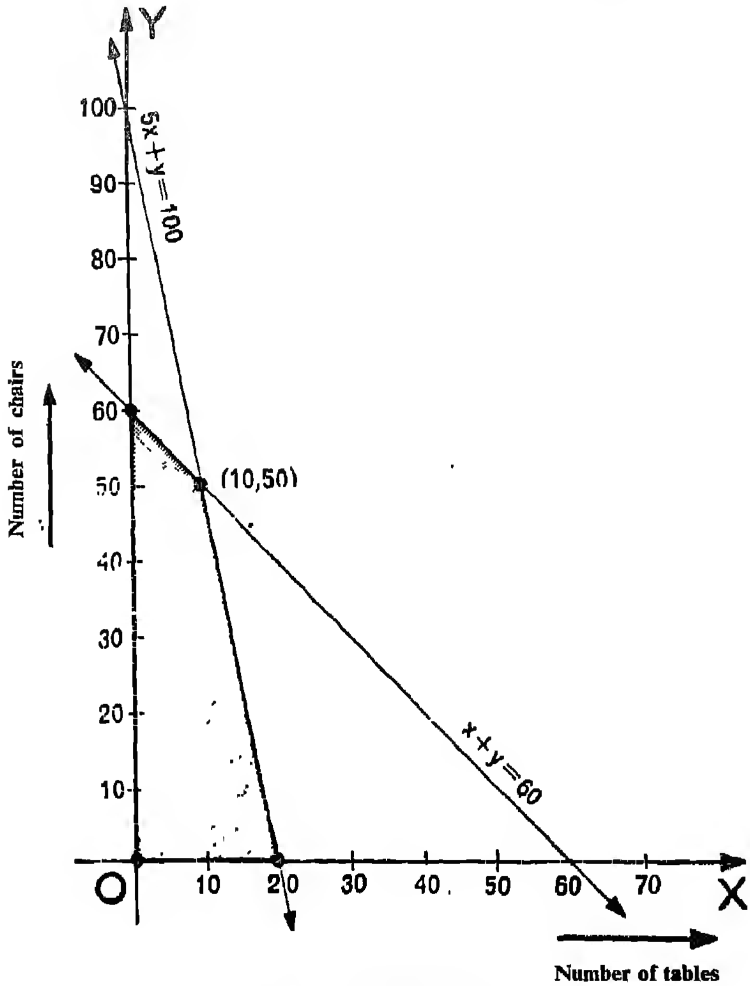


Fig. 22.1 : Feasible region

dealer for investing in tables and chairs. The region, therefore, is called a **feasible region***. Every point of this region is called a **feasible solution to the linear programming** problem.

*The intersection of a finite number of closed half-planes is called a **polygonal convex set**. Feasible region is, therefore, always a **convex polygon**.

It is a matter of commonsense that the dealer would seek only that (those) feasible solution(s) which would maximize his profit, namely, the objective function stated as equation (5) What we could, therefore, do is make a table of P-values for different feasible choices and select the feasible solution(s) which yield maximum P. This, however, would usually involve enormous and unnecessary calculations As an alternative, we use the graphical method to find the maximum(s) of the objective function.

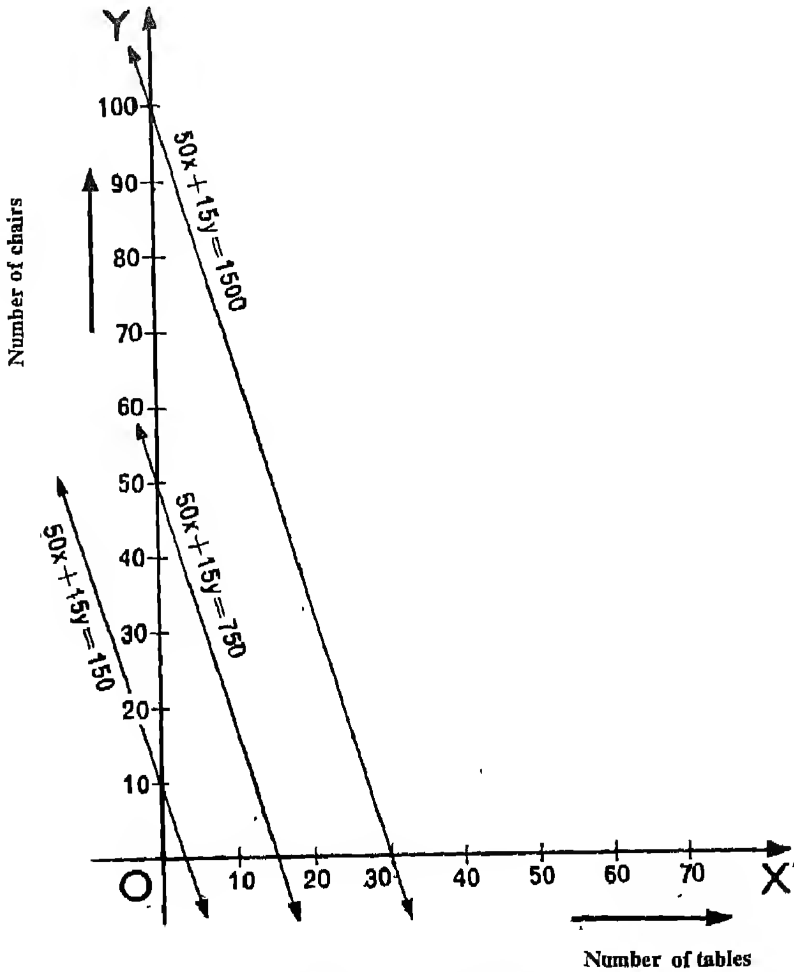


Fig. 22.2 : Isoprofit lines

At first sight, the reader may be a bit puzzled as to how to graph the profit function since it is not in the form of an equation. This, however, should cause no concern. All we need to do is to take an arbitrary point, preferably in the feasible region and deter-

mine P. For instance, let us take the point (3,0). P, therefore, is Rs 150 00 (Why ?) It is now easy to graph

$$50x+15y=150$$

The graph, of course, is a straight line. Further, every point on this line yields a profit of Rs 150.00. No wonder, it is called an **isoprofit line**. (See Fig. 22.2). Similarly, we can draw other isoprofit lines with respect to other values of P obtained by taking different points.

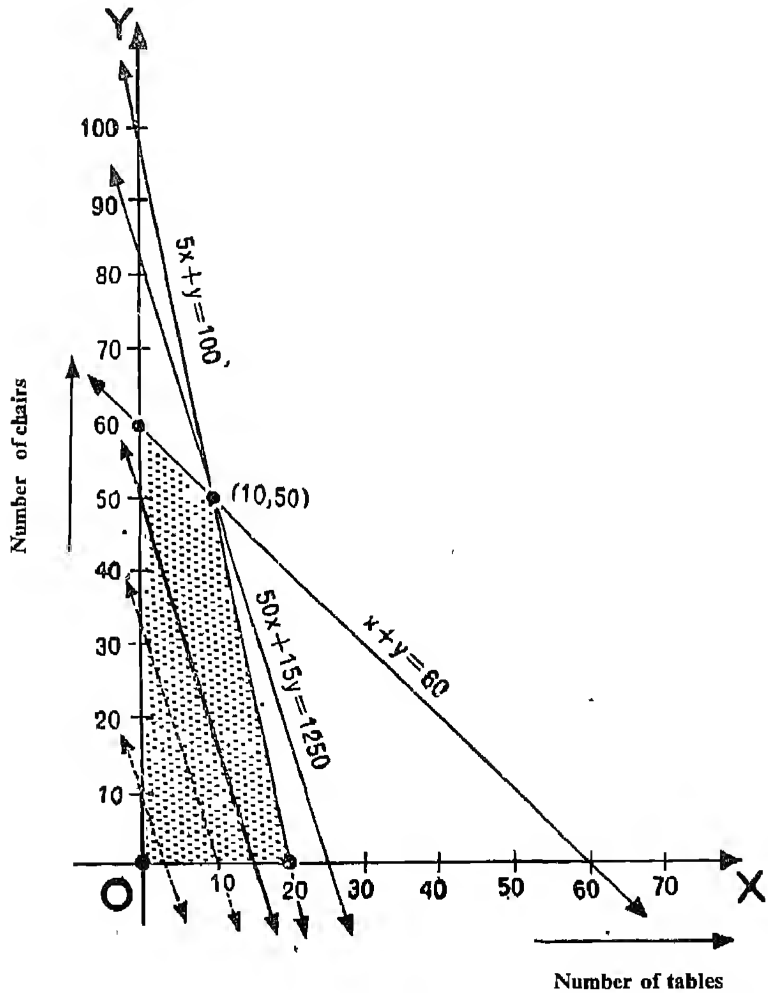


Fig 22 3

The objective function, therefore, represents a set of parallel lines for different values of P. (Why ?). The line $150=50x+15y$ is a member of this set. We observe

that the farther the line from the origin, the greater is the value of P. Commonsense dictates, we should continue to draw isoprofit lines to realize higher profits until the line is in a position where it has at least one point in common with the points in the set of feasible solutions and from which position if the line is moved further in the direction away from the origin, the line will not have any point in common with the points in the set of feasible solutions. The feasible solution(s) corresponding to this position of the line gives the maximum* value(s) of the function. (See Fig. 22.3)

We observe from Fig. 22.3 that the desired position of the isoprofit line is obtained at the vertex** (10,50) of the region

The optimal investment strategy for the dealer would be, therefore, to invest in 10 tables and 50 chairs. Corresponding to this strategy, his profit would be maximum, namely, Rs 1250.00. (Why ?)

22.5 Some Remarks on the Graphical Method of Solving Linear Programming Problems

Remark 1 : Since the maximum (or minimum) occurs only at the boundary point(s), we need to consider only those positions of the isoprofit line which pass through the corner(s) of the feasible region.

Remark 2 : In case the isoprofit line is parallel to any of the boundary lines of the region, the maximum (or minimum) will occur only at the points of that boundary line.

Remark 3 : Let us calculate the values of P at each of the vertices of the feasible region in the Example of Section 22.2 and represent them in the table below :

alues of P Corresponding to the Vertices of the Feasible Region

Vertex of the Feasible Region	*Corresponding value of P (in rupees)*
(0, 0)	0.00
(0, 60)	900.00
(10, 50)	1250.00←**Maximum Profit**
(20,0)	1000.00

*The Fundamental Extreme Point Theorem assures us that the maximum (or minimum) of the objective function will occur only at the boundary point (s).

**A vertex or corner is the point of intersection of two or more boundary lines.

We observe that the maximum profit to the dealer results from the investment strategy (10,50).

22.6 Some More Examples on Linear Programming

Example 1 : A dietician wishes to mix two types of foods in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C Food '1' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food '2' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 5.00 per kg to purchase food '1' and Rs 7 00 per kg to purchase food '2' Determine the minimum cost of such a mixture.

Solution · Let the dietician mix x kg of food '1' and y kg of food '2' Clearly,

$$x \geqslant 0 \tag{1}$$

$$y \geqslant 0 \tag{2}$$

Since the mixture must contain at least 8 units of vitamin A and 10 units of vitamin C, we have the constraints

$$2x+y \geqslant 8 \tag{3}$$

and

$$x+2y \geqslant 10 \tag{4}$$

The objective function or the cost C of the mixture is

$$C = 5x+7y \tag{5}$$

Mathematically, therefore, the problem reduces to minimizing (5) subject to the constraints (1), (2), (3) and (4). As before, we will attempt a graphical solution of this problem. Let us refer to Fig. 22.4

The feasible region is the shaded region in the figure. (0, 8), (2, 4) and (10, 0) are its corners (Why ?)* For different values of C, we will obtain a set of parallel lines Of course, each line, in this case, would be an **isocost line.** We observe that the nearer the line to the origin, the smaller is the value of C. We need to concern ourselves with only those positions of the isocost line which pass through the corners of the feasible region. The desired position of the isocost line is obtained at the vertex (2, 4) of the region.

The optimal mixing strategy for the dietician would be, therefore, to mix 2 kg of food '1' and 4 kg of food '2' Corresponding to this strategy, his cost would be minimum, namely, Rs 38.00 (Why ?)

---

*(0, 8) and (10, 0) are obvious. (2, 4) is the point of intersection of the two boundary lines $2x+y=8$ and $x+2y=10$. The reader is advised to verify for himself by solving the two equations simultaneously.

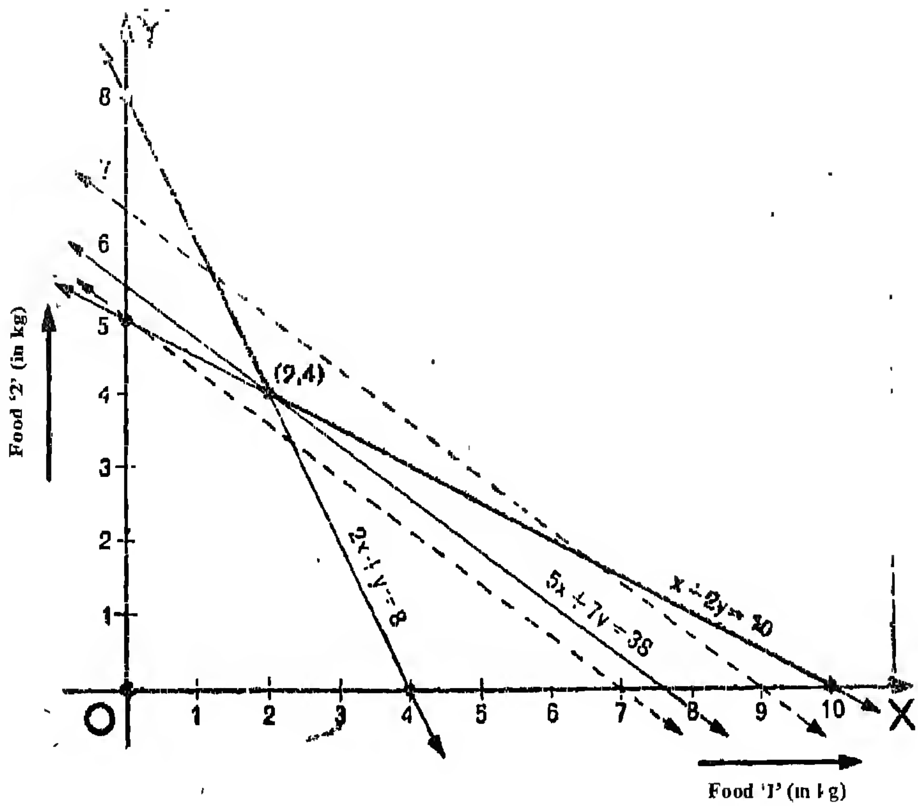


Fig. 22 4

Example 2 : A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs 2.50 per package on nuts and Re 1.00 per package on bolts. How many packages of each should he produce each day so as to maximize his profit, if he operates his machines for at the most 12 hours a day ?

Solution : Let the manufacturer produce x packages of nuts and y packages of bolts each day. Clearly,

$$x \geqslant 0 \tag{1}$$

$$y \geqslant 0 \tag{2}$$

Since the machines can operate for at most 12 hours a day, we have the constraints corresponding to machines A and B, respectively, as

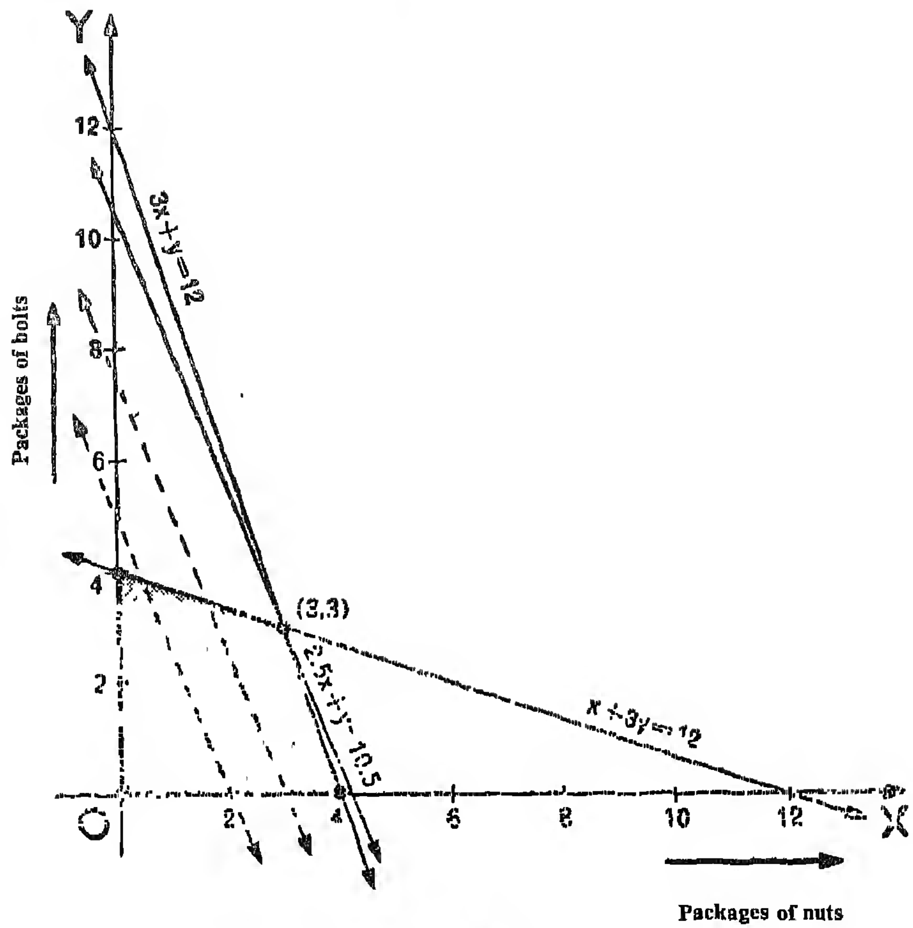


Fig. 22.5

$$x+3y \leqslant 12 \tag{3}$$

and $$3x+y \leqslant 12 \tag{4}$$

The objective function or the profit, P, is

$$P=2.5x+y \tag{5}$$

Mathematically, therefore, the problem reduces to maximizing (5) subject to the constraints (1), (2), (3) and (4). Let us refer to Fig. 22.5.

The feasible region is the shaded region in the figure. (0, 0), (0, 4), (3, 3) and (4,0) are its corners. For different values of P, we obtain a set of parallel isoprofit lines. If we confine ourselves to only those positions of the isoprofit line which pass through the

corners of the feasible region, we observe that the isoprofit line through the vertex (3, 3) yields the maximum profit

The optimal production-strategy for the manufacturer would be, therefore, to manufacture 3 packages each of nuts and bolts daily Corresponding to this strategy his profit would be maximum, namely, Rs 10.50 (Why ?)

***Example 3 : (Transportation Problem—A Simplified Version)**

There is a factory located at each of the two places P and Q From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C The weekly requirements of the depots are, respectively, 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are, respectively, 8 and 6 units. The cost of transportation per unit is given below

To / *From*	*Cost (in rupees)*		
	A	*B*	*C*
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum ?

Solution · Let x and y units be transported from the factory at P to the depots at A and B, respectively. Then $(8-x-y)$ units will be transported from the factory at P to the depot at C. (Why ?) Clearly

$$x \geqslant 0 \qquad (1)$$

$$y \geqslant 0 \qquad (2)$$

and

$$8-x-y \geqslant 0 \qquad (3)$$

The weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory, at P, the remaining $(5-x)$ units need to be transported from the factory at Q. Obviously

$$(5-x) \geqslant 0 \qquad (4)$$

Similarly, $(5-y)$ and $[4-(8-x-y)]$ units need to be transported from the factory at Q to the depots at B and C respectively. Also

$$(5-y) \geqslant 0 \tag{5}$$

and $$[4-(8-x-y)] \geqslant 0$$

i e., $$(x+y-4) \geqslant 0 \tag{6}$$

The objective function or the transportation cost T is

$$T=16x+10y+15(8-x-y)+10(5-x)+12(5-y)+10(x+y-4)$$

Or, $$T=x-7y+190 \tag{7}$$

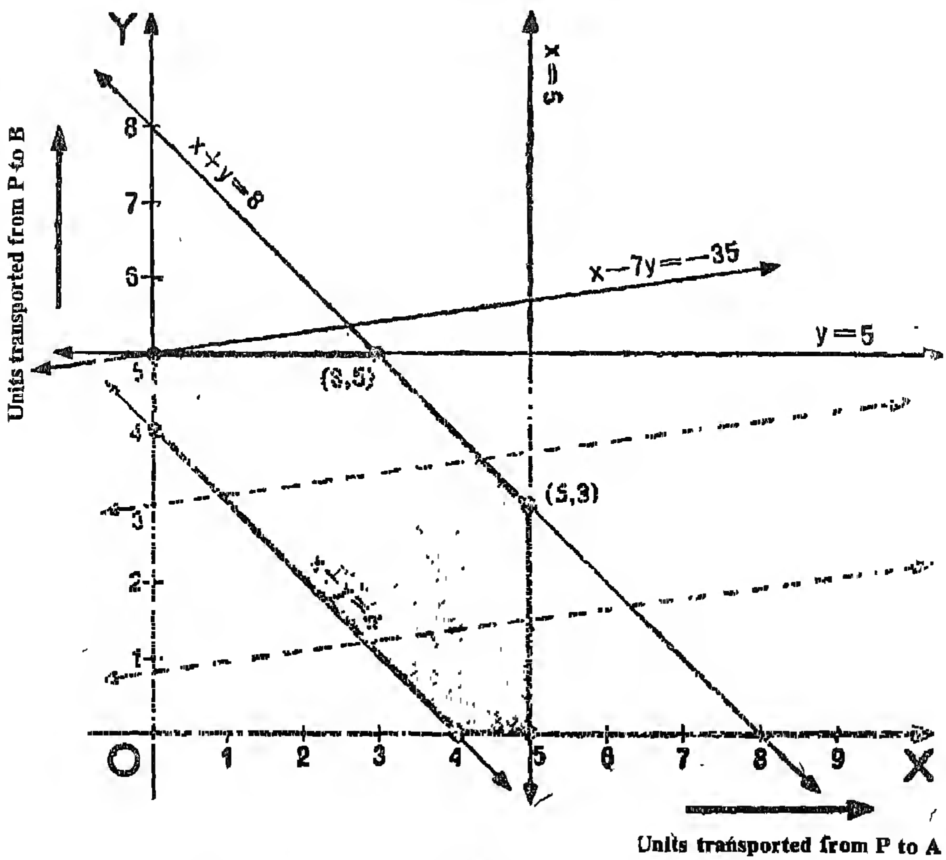


Fig. 22.6

Mathematically, therefore, the problem reduces to minimizing (7) subject to the constraints (1), (2), (3), (4), (5) and (6). Let us refer to Fig. 22.6.

(0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0), are the corners of the feasible region For different values of T, we obtain a set of parallel isocost lines. Again, confinir

ourselves to only those positions of the isocost line which pass through the corners of the feasible region, we observe that the isocost line through the vertex (0, 5) yields the minimum cost

The optimal transportation strategy would be, therefore, to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C, respectively. Corresponding to this strategy, the transportation cost would be minimum, namely, Rs 155.00 (Why ?)

[The reader is advised to construct a table of values of T corresponding to each vertex of the feasible region and verify, as in Remark 3, Section 22.5.]

EXERCISE 22.1

1. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs 5760 00 to invest and has space for at most 20 items. A fan costs him Rs 360.00 and a sewing machine Rs 240.00 His expectation is that he can sell a fan at a profit of Rs 22.00 and a sewing machine at a profit of Rs 18.00

   Assuming he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ?

2. Maximize $P=3x+4y$ subject to the constraints

$$x \geqslant 0$$
$$y \geqslant 0$$
$$\text{and} \quad x+y \leqslant 4$$

3. Minimize $C=3x+y$ subject to the costraints

$$x \geqslant 0$$
$$y \geqslant 0$$
$$x+y \leqslant 2$$
$$\text{and} \quad y+4x \leqslant 5$$

*4. A manufacturer has 3 machines installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must operate at least 5 hours a day. He produces only two items, each requiring the use of the three machines. The number of hours

required for producing 1 unit each of the items on the three machines is given in the following table :

Item	Number of hours required by the machine		
	I	II	III
A	1	2	1
B	2	1	$\frac{5}{4}$

He makes a profit of Rs 6 00 on item A and Rs 4 00 on item B. Assuming he can sell all that he produces, how many of each item should he produce so as to maximize his profit ? Determine his maximum profit

5. Two tailors A and B earn Rs 15.00 and Rs 20.00 per day respectively. A can stitch 6 shirts and 4 pants while B can stitch 10 shirts and 4 pants per day How many days shall each work if it is desired to produce (at least) 60 shirts and 32 pants at a minimum labour cost ?

6. A factory manufactures 2 types of screws, A and B, each type requiring the use of two machines—an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machine to manufacture a package of screws 'A', while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a package of screws 'B' Each machine is available for at most 4 hours on any day. The manufacturer can sell a package of screws 'A' at a profit of 70 p and screws 'B' at a profit of Re 1.00.

   Assuming he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day, in order to maximize his profit ? Determine the maximum profit.

7. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp while is takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at most 20 hours and the grinding/cutting machine for at most 12 hours. The profit from the sale of a lamp is Rs 5.00 and of a shade is Rs 3 00

   Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximize his profit ?

*8 Two godowns A and B have a grain-storage capacity of 100 quintals and 50 quintals, respectively They supply to 3 ration shops D, E and F whose requirements are 60, 50 and 40 quintals, respectively. The costs of transportation per quintal from the godowns to the shops are given in the following table.

Transportation costs per quintal (in rupees)		
	From	
To	*A*	*B*
D	6.00	4.00
E	3 00	2 00
F	2.50	3 00

How should the supplies be transported in order that the transportation cost is minimum ?

*9 An oil company has two depots, A and B, with capacities of 7000 litres and 4000 litres, respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500, 3000 and 3500 litres, respectively. The distance (in km) between the depots and the petrol pumps is given in the following table.

Distance (in km)		
	From	
To	*A*	*B*
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost/km is 1 paisa per litre, how should the delivery be scheduled in order that the transportation cost is minimum ?

22.7 Historically Speaking

The reader must have realized by now that the mathematical formulation of the linear programming problems invariably leads to a system of linear equations/inequations

and polygonal convex sets The theory of linear inequalities and convex sets has developed extensively in the past hundred years. However, the field of linear programming problems is fairly recent in its origin.

The first problem in linear programming was formulated in 1941 by the Russian mathematician **L. Kantorovich** and the American economist **F.L. Hitchcock**, both of whom worked at it independently of each other This is the well-known **transportation problem.** In 1945, an English economist, **G. Stigler**, described yet another linear programming problem—that of determining an optimal diet

In the Second World War, when the war operations had to be planned to economize on expenditure, minimize losses and maximize damage to the enemy, linear programming problems came to the forefront However, they remained top-secret until after the war, when in 1947, an American economist, **G.B. Dantzig** published a paper in the famous journal *Econometrica* wherein he formulated the general linear programming problem **George Dantzig** is also credited with using the term 'Linear Programming' and for the solution of the problem by analytical methods

Linear programming is now widely used in planning economic activity and is an accepted tool in the formulation of national plans

For his work on these problems, the Russian mathematician, **L. Kantorovich**, was awarded Nobel Prize for Economics, in the year 1974, together with another famous (American) mathematical-economist, **T.C. Koopmans**

22.8 Key Concepts

Objective function	Polygonal convex set
Linear constraints	Convex polygon
Feasible region	The Fundamental Extreme Point Theorem
Feasible solution(s)	Transportation problem

22.9 Suggestions for Further Reading

For an excellent introduction to inequalities, the reader is referred to the paper back

[1] Edwin Beckenbach and Richard Bellman **An Introduction to Inequalities.** The L.W Singer Company, New York (U S A) 1961

A delightful book for the beginners in Linear Progamming is

[2] N Paul Loomba . **Linear Programming-An Introductory Analysis.**
Mcgraw Hill Book Company, New York. 1964.

A somewhat advanced treatment on Linear Programming is found in

[3] G Hadley : **Linear Programming**
Addison Wesley Publishing Company, Inc., Massachusetts (U.S.A.). 1962

***For** an introduction to analytical tools for studying linear programming **problems**, the reader is referred to*

[4] F.A. Ficken : **The Simplex Method of Linear Programming.**
Holt. Rinehart and Winston, New York (U.S A). 1967

A reference for the serious student of Linear Programming is

[5] Dantzig, G.B. : **Linear Programming.**
Princeton University Press, New Jersey (U S.A.) 1963.

UNIT XXIII

ARITHMETICAL DESCRIPTORS OF DATA

We learn how to find the mean, median, mode, standard deviation and variance of the grouped data

23.1 Recall

You should review the following concepts from your earlier classes. In particular, you should recall,

Methods of collection of data—primary and secondary data

Raw or ungrouped data

Grouped data and frequency distributions—classes, class-interval or class-size, lower and upper limits of a class, tally marks, class frequency and class-mark.

Cumulative frequency distributions—cumulative frequency

Graphical representation of data—bar charts, histograms, frequency polygons, cumulative frequency graphs or ogives, pie charts and pictographs.

Mean of raw data, M or $\bar{x}$

Mean of grouped data, M_g or $\bar{y}$

We also recall that **Statistics is the science which enables us to draw representative samples, analyze the data collected, interpret them and make inferences about the populations.**

23.2 Retrospect

We must point out that we have not and shall not concern ourselves with the techniques of drawing representative samples. Although 'proper' sampling is very much

the concern of a statistician, the study of the techniques comprises, in itself, a branch of Statistics called the **Theory of Sampling**. If we were to do any justice, we would need to devote a complete volume to the subject of sampling We **will, therefore, simply assume that the sample data that we study are from representative sample.**

Before we are able to draw inferences from the data, we must '**know**' our data well When there are too many observations, we **condense** the data by grouping them into frequency distributions to bring out some of their salient features We also **represent them pictorially** to make their salient features visually more perceptive. We '**describe**' them **arithmetically** in that we calculate some (arithmetical) numbers to represent certain features of the data We have discussed only one arithmetical descriptor so far, namely, the mean, although we did mention in passing about the median and the mode

Mean of raw data is simple to calculate. If $x_1, x_2, x_3, \ldots, x_n$ **are the n observations, we recall that**

$$M(\text{or } \bar{x}) = \frac{\sum_{i=1}^{n} x_i}{n}$$

For calculating the **mean of the grouped data, we first have to make an assumption,** namely, **the frequencies in each class are centered at its class-mark.** Then, if y_i's denote the **class-marks and f_i's the class-frequencies, we have**

$$M_g\ (\text{or } \bar{y}) = \frac{\sum_{i=1}^{k} f_i y_i}{n}$$

where k is the number of classes and n the number of observations. We recall that

$$n = f_1 + f_2 + \ldots + f_k \text{ i.e., } n = \sum_{i=1}^{k} f_i$$

Of course, if y_i's are large, this formula presents a problem We had, therefore, suggested an **alternative formula for calculating the mean** or what is commonly known as the **deviation method** of calculating the mean. Do you recall this method ?

We let one of the class-marks be arbitrarily* designated as 'a' and take deviations $d_i = y_i - a$.

$$\text{Then } M_g\ (\text{or } \bar{y}) = a + \frac{1}{n} \sum_{i=1}^{k} f_i d_i$$

*Computations are further simplified if 'a' is chosen somewhere near the middle.

We now give a mathematical justification for this short-cut method.

We have, $d_i = y_i - a$

Thus, $f_i d_i = f_i y_i - f_i a$

If we calculate $(f_i d_i)$'s for each class and sum, we get

$$\sum_{i=1}^{k} f_i d_i = \sum_{i=1}^{k} f_i y_i - a \sum_{i=1}^{k} f_i \qquad (1)$$

Divide both sides of (1) by n. We have

$$\frac{1}{n} \sum_{i=1}^{k} f_i d_i = \frac{1}{n} \sum_{i=1}^{k} f_i y_i - \frac{1}{n}(an) \qquad (2)$$

But $\frac{1}{n} \sum_{i=1}^{k} f_i y_i = M_g$. Thus (2) becomes

$$M_g = a + \frac{1}{n} \sum_{i=1}^{k} f_i d_i$$

23.3 Mean of the Grouped Data Again

Since in most of the frequency distributions that we shall be concerned with, the classes are equal, **we can further simplify the calculation of the mean of the grouped data.**

Let us denote by **c, the class-size.**

Now we not only take the deviation of each class-mark from the arbitrarily **chosen** 'a', but also divide each deviation by c. Let

$$u_i = \frac{y_i - a}{c} \qquad (1)$$

Then

$$f_i u_i = \frac{1}{c}\left[f_i y_i - a f_i \right]$$

If we calculate $(f_i u_i)$'s for each class **and sum over** k **classes, we get**

$$\Sigma f_i u_i = \frac{1}{c} \Sigma f_i y_i - \frac{a}{c} \Sigma f_i \qquad (2)$$

We divide both sides of (2) by n. We have

$$\frac{1}{n}\Sigma f_i u_i = \frac{1}{c}\frac{\Sigma f_i y_i}{n} - \frac{a}{nc}(n) \tag{3}$$

But $\frac{1}{n}\Sigma f_i y_i = M_g$ Thus (3) becomes

$$M_g = a + \frac{c}{n}\sum_{i=1}^{k} f_i u_i \tag{4}$$

If we denote by M_u, the mean of the u_i's (sometimes, called the **coded mean**), (4) can be written as

$$M_g = a + cM_u \tag{5}$$

Some authors refer to this method as the **step deviation method.**

We now consider some examples.

Example 1 : The weekly observations on Cost of Living Index in a certain city for the year 1970-71 are given below :

Cost of Living Index	*Number of Weeks*
140-150	5
150-160	10
160-170	20
170-180	9
180-190	6
190-200	2
TOTAL	52

Compute the average weekly Cost of Living Index

[Also see Section 14.4 and 14.5 of Mathematics—A Text Book for Secondary Schools, Part II, 2nd edn , 1978 ; National Council of Educational Research and Training, New Delhi.]

Solution : We note that

$$k=6,\ n=\Sigma f_i=52 \text{ and } c=10$$

We let $\quad a=y_3=165$

We then make the following columnar calculations :

Calculation of the Average Weekly Cost of Living Index

Cost of Living Index	*Number of Weeks* (f_i)	y_i	$u_i = \frac{y_i - 165}{10}$	$f_i u_i$
140-150	5	145	-2	-10
150-160	10	155	-1	-10
160-170	20	165	0	0
170-180	9	175	$+1$	$+9$
180-190	6	185	$+2$	$+12$
190-200	2	195	$+3$	$+6$
TOTAL	52			$+7$

$$M_u = \frac{1}{n} \Sigma f_i u_i = \frac{7}{52}$$

Thus $$M_g = a + cM_u$$

$$= 165 + 10\left(\frac{7}{52}\right)$$

$$= 166.3 \text{ (approx.)}$$

Hence, the average weekly Cost of Living Index is 166.3 (approx.).

Example 2 : The melting points in °F of 45 common chemical elements are given in the table below :

Melting Point (in °F)	*Number of Common Chemical Elements*
$-499-0$	10
$1-500$	8
$501-1000$	4
$1001-1500$	6
$1501-2000$	5
$2001-2500$	2
$2501-3000$	4
3001 and higher	6
TOTAL	45

Find the average melting point in °F.

Solution : We note that there is one open-ended class, namely, 3001 and higher. In the absence of any other information, we assume that the class interval '3001 and higher' is '3001-3500'. (Why ?) Then $c=500$

Also, $k=8$ and $n=45$

We let $a=1750.5$. The following columnar calculations are done to determine M_g

Calculation of the Average Melting Point in °F of 45 Common Chemical Elements

Melting Point in °F	f_i	y_i	$u_i=\frac{y_i-1750.5}{500}$	f_iu_i
−499−0	10	−249 5	−4	−40
1−500	8	250.5	−3	−24
501−1000	4	750 5	−2	− 8
1001−1500	6	1250 5	−1	− 6
1501−2000	5	1750.5	0	0
2001−2500	2	2250.5	+1	+ 2
2501−3000	4	2750.5	+2	+ 8
3001−3500	6	3250 5	+3	+18
TOTAL	45			−50

Thus, $$M_u=\frac{-50}{45}=\frac{-10}{9}$$

Hence, $$M_g=a+cM_u=1750.5+500\left(\frac{-10}{9}\right)$$
$$=1750.5-555\ 6$$
$$=1194\ 9 \quad \text{(approx.)}$$

Therefore, the average melting point of 45 common chemical elements is, approximately, 1194.9 °F.

13.4 Median and Mode

13.4.1 Median of Raw Data

Median, we recall, is the value of the middlemost observation, when the data are arranged in ascending or descending order of magnitude. To calculate the median of raw data, therefore, we go through the following steps :

Step 1 : We arrange the data in increasing or decreasing order.

Step 2 : If n, the number of observations, is odd then median is the value of the $\left(\frac{n+1}{2}\right)$th observation.

If n, however, is even, the median is taken as the average of the $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

We consider some examples.

Example 1 The marks, out of 50, of a certain class consisting of 15 students, in a class-test are given below

18, 19, 25, 29, 24, 23, 32, 40, 19, 26, 22, 20, 18, 35, 21

Find the median score.

Solution We first arrange the marks in, say, ascending order.

We have

18, 18, 19, 19, 20, 21, 22, 23, 24, 25, 26, 29, 32, 35, 40

Thus the median score is the value of $\left(\frac{15+1}{2}\right)$th or 8th observation. The median score is, therefore, 23.

[Note The median is **not 8. It is the value of the 8th observation.**]

Example 2 The daily earnings (in Rs) of 20 workers in a certain factory are

16, 8, 19, 7, 12, 6, 13, 14, 16, 17, 19, 5, 20, 9, 12, 10, 8, 11, 16, 13

Find the median earning.

Solution : We first arrange the wages, in say, ascending order. We have

5, 6, 7, 8, 8, 9, 10, 11, 12, 12, 13, 13, 14, 16, 16, 16, 17, 19, 19, 20

Since there is an even number of observations, we take the median as the average of the $\left(\frac{20}{2}\right)$th and $\left(\frac{20}{2}+1\right)$th, *i e* , 10th and 11th observations. The value of the 10th observation is 12 and that of the 11th observation is 13 The median is, therefore, the average of these two observations, *i e.*,

$$\frac{12+13}{2}=12.50$$

In other words, the median earning is Rs 12.50.

23.4.2 Mode of Raw Data

Mode, we recall, is **the value of the observation that occurs most often**. It is **rather** trivial to find mode of raw data. We simply scan the observations and note which **observation** occurs most often. Consider, for instance, the example of daily earnings of 20 workers in a certain factory [See Example 2, Section 23.4.1]. The daily earnings (in Rs) are

16, 8, 19, 7, 12, 6, 13, 14, 16, 17, 19, 5, 20, 9, 12, 10, 8, 11, 16, 13

We **observe** that 16 occurs most often. We say that the **modal earning or mode is Rs 16**.

It is possible to have more than one **mode**. Consider, for instance, **the** example of the **marks** out of 50 of 15 students in a certain class-test [See Example 1, Section 23.4.1]. **The marks** are

18, 19, 25, 29, 24, 23, 32, 40, 19, 26, 22, 20, 18, 35, 21

We observe that both the observations 18 and 19, occur most often. We say there are two **modal marks** or **modes**, namely, 18 and 19.

What if each observation occurs only once or equally often? **We say that the data have no mode.**

23.4.3 Median of Grouped Data

We now learn how to find the **median of grouped data.** Let us consider an example.

Example 3. Following is the distribution of rents in a certain city for a 2-room set, based on a sample of size 100.

Rent (in Rs)		*Number of 2-Room Sets* (f_i)	*Cumulative Frequency* $(c.f.)$
150—175		10	10
175—200		13	23
200—225		17	40 } $j=\frac{100}{2}-40$
225—250	**Median Class**	**15**	55 }
250—275		16	71
275—300		10	81
300—325		7	88
325—350		5	93
350—375		4	97
375—400		3	100
TOTAL		100	

Compute the median rent.

Solution : We wish to compute the median rent, *i.e*, the amount (of rent) below which 50% and above which 50% of the 2-room sets in the city are rented

First we must make an assumption regarding the frequencies in each class What should a 'reasonable' assumption be ? We assume* **that the frequencies in each class are distributed evenly (uniformly) throughout the class.** We go through the following steps :

Step 1 · Make a column of cumulative frequencies

Step 2 Determine the **median-class,** *i.e.*, **the class in which the** $\left(\frac{n}{2}\right)$**th observation lies.**

$\Big[$In our example, $n=100$. Thus $\left(\frac{100}{2}\right)$th observation will lie in the class 225—250 $\Big]$

We now want to find the point in the median-class at which the $\left(\frac{100}{2}\right)$th observation lies To arrive at the $\left(\frac{100}{2}\right)$th or 50th observation, we need $50-(10+13+17)$ $=50-40=10$ of the 15 observations that are assumed to be uniformly distributed throughout the median-class. We use the unitary method and get the required point as

$$225+\frac{\left(\frac{100}{2}-40\right)}{15}\times 25$$

$$=225+16.67$$

$$=241.67 \text{ (approx.)}$$

In terms of steps, we have

Step 3 . Calculate $\mathbf{j}=\frac{n}{2}-$**c. f of Med** $_{-1}$ **class****

---

*The reader may wonder why we did not make the same assumption that we made for calculating the mean of the grouped data. A moment's reflection will show that the value of the median, with that assumption, will be 'cruder' than the one we obtain with the assumption which we made.

**Med$_{-1}$ class is an abbreviation to denote the class immediately preceeding the median class.

Step 4 Then

$$\text{Med} = L_{med} + \frac{j\,c}{f_{med}}$$

where L_{med} = lower limit* of the median-class

f_{med} = frequency of the median-class

c = class-size

and where j is as defined in Step 3.

Thus 50 percent of the 2-room sets in this sample rent out at (approx.) Rs 241.67 or less and 50 percent at (approx.) Rs 241.67 or more

[Had we assumed that the frequencies in each class are centred at its class-mark, the median would be Rs 237 50.]

We see that the median provides us with a dividing line to separate the higher from the lower values

We consider some more examples,

Example 4 The marks (out of 75) obtained by 60 students in a certain examination are given below

Marks	*Number of Students*	*c. f.*
15—20	4	4
20—25	5	9
25—30	11	20
30—35	6	26
35—40 ← Median Class	5	**31** $\left.\begin{matrix}26\\31\end{matrix}\right\\} j=\frac{60}{2}-26$
40—45	8	39
45—50	9	48
50—55	6	54
55—60	4	58
60—65	2	60
TOTAL	60	

Calculate the median mark.

*Assuming there are no gaps between the upper limit of a class and the lower limit of the class immediately following it.

Solution : We go through the **four** steps. We make a column of **cumulative** frequencies

Now, $\frac{n}{2}=\frac{60}{2}=30$

Thus, median-class is 35—40

And, $j=\frac{n}{2}$ − c. f. of Med − 1 class

$=30-26=4$

$L_{med}=35$

$c=5$

$f_{med}=5$

Thus, $\text{Median}=35+\frac{4\times5}{5}=39$

Hence, the median score is 39 In other words, half the class scored 39 marks or less (out of 75) and the other half scored 39 marks or more (out of 75)

23.4.4 Mode of Grouped Data

As we have seen, mode, the value that occurs most often may exist but may not be unique or may not even exist We will, therefore, not spend much time on learning how to calculate mode of grouped data, except very crudely

We find the **modal class,** namely, the class **corresponding to maximum frequency.** [There may be more than one modal class.] We then have to make a **'reasonable'** assumption **We may assume that the frequencies in each class are centred at its class-mark. In that case, the mid-point of the modal class is the mode.**

We refer to Example 3, Section 23 4 3 We observe that the modal class is 200—225

Thus, Mode=212.50

i e., the modal rent is Rs 212.50 Of course, the modal frequency is 17 We now refer to to Example 4, Section 23.4.3. We observe that the modal-class is 25--30

Thus, **Mode=27 5**

i.e , the modal score (mark) is 27 5. Of course, the modal frequency is 11

23.4.5 Empirical Relation Between Mean, Median and Mode

We will learn later that in a symmetrical (bell-shaped) distribution, mean, median and mode coincide and that in other distributions, they do not In a majority of other distributions that we come across, we find that

Mode=Mean—3 (Mean—Median)

In such distributions, we can sometimes use the formula to calculate the mode. Of course, the values of the mode calculated by the method of Section 23.4 4 and by using the Empirical Relation will clearly be different. As already remarked, we are only concerned with 'crude' calculations of mode and will, therefore, not bother about the difference in the two values Let us consider Example 3, Section 23 4.3. The median rent is Rs 241 67 The mean rent is Rs 248 00 [See Question 11, Exercise 23.1].

Thus, Mode=248 0—3 (248.0—241.67)

=229.01

[The reader is advised to draw a histogram to represent the distribution of rents and plot the mean, median and mode on the graph]

EXERCISE 23.1.

1. Given below is a part of the Fibonacci sequence :

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 .

Find the mean of these 11 numbers. Verify that

$$\sum_{i=1}^{11} (x_i - \bar{x}) = 0$$

2. Find the average of first n natural numbers.

3. Find the mean of the squares of first n natural numbers.

4. Following is the distribution of the number of outdoor patients registered in a certain hospital in 180 days :

Number of Patients	101—150	151—200	201—250	251—300	301—350	351—400	401—450
Number of Days	7	10	47	72	25	11	8

(*i*) Compute the average number of outdoor patients registered per day. State the assumption that you make to calculate the average.

(*ii*) What assumption would you make to calculate the mode of the above data? State the assumption and calculate the mode

(*iii*) Draw a histogram to represent the above data. Plot the average and the mode.

5. Daily earnings of 600 workers in a factory are given below

Daily Earnings (in Rs)	8—10	10—12	12—14	14—16	16—18	18—20	20—22	22—24
No. of Workers	20	25	30	36	35	100	70	90

Daily Earnings (in Rs)	24—26	26—28	28—30	30—32	32—34	34—36	36—38	38—40
No of Workers	41	40	35	40	20	10	5	3

(*i*) Compute the average daily earnings.

(*ii*) Calculate also the median and the mode. State the assumptions that you make to calculate the median and the mode.

(*iii*) Draw a frequency polygon to represent the above data. Plot the mean, median and mode.

6. Following is the distribution of the distance travelled, in million kilometres, by the Indian Railways for 22 years from 1952-53 to 1973-74.

Distance Travelled (in million kms)	300—320	320—340	340—360	360—380	380—400
Number of Years	2	2	1	3	2

Distance Travelled (in million kms)	400—420	420—440	0—460	460—480
Number of Years	1	3	3	5

Determine the mean distance travelled by the Indian Railways in a year.

[Source : Raw data from Yojna ; Vol. XIX ; No. 3. 1st March, 1975, Page 14]

7. Weights of trainees in a wrestling coaching camp are given in the following table. Calculate the average weight.

Weight (in kg)	Below 40	40—45	45—50	50—55	55—60	60—65	65—70	70—75	75—80
Number of Trainees	3	5	6	10	12	10	8	4	2

State any assumption that you make in calculating the average.

8. Gross cultivated areas, in million hectares, in 22 States and Union Territories of India, are given below :

12488, 2600, 10770, 10137, 5156, 898, 815, 10413, 2685, 19637, 19195, 188, 174, 49, 7454, 5435, 16624, 6745, 335, 22668, 6528, 422

Determine the mean, median and mode.

[Source : Yojna ; Vol. XX ; No. 1, 26th January 1976, Page 24]

9. Determine the mean gain in weight for 25 children in a school during a specified period.

Gain in Weight (in kg)	6.0	5.5	5.0	4.5	4.0	3.5	3.0	2.5	2.0
Number of Children	1	3	4	1	5	2	4	3	2

10. Given (on the next page) is a frequency distribution of auditory reaction times for 188 students at the University of Chicago, U.S.A.

Auditory Reaction Times (in sec)	Frequency
.14—.15	1
.16—.17	12
.18—.19	37
.20—.21	69
.22—.23	28
.24—.25	17
.26—.27	11
.28—.29	5
.30—.31	4
.32—.33	2
.34—.35	2
TOTAL	188

Find the average auditory reaction time of the group.

[Source : L.L Thurston, A Factorial Study of Perception, University of Chicago Press, Chicago. 1944.]

11. Rent in a certain city for a 2-room set are given below :

Rent (in Rs)	Number of 2-room Sets
150—175	10
175—200	13
200—225	17
225—250	15
250—275	16
275—300	10
300—325	7
325—350	5
350—375	4
375—400	3

Compute the average rent.

12. In a survey of 950 households in a village, the following distribution of the number of children per household is obtained :

No. of Children	0—2	2—4	4—6	6—8	8—10	10—12
No of Households	272	328	205	120	15	10

Find the median and modal number of children per household. Find also the modal frequency

13. The mean of 10 numbers is 7 and the mean of 15 other numbers is 12. Determine the mean of 25 numbers taken together.

14. The median of 10 observations is 7 and the median of 15 other observations is 12. What can you say about the median of 25 observations taken together ?

15. The mean of 18 observations is -7 If each observation is increased by 3, determine the mean of the new set.

16. The mean of 12 observations is 6. If each observation is multiplied by 2, determine the mean of the new set.

17. Scores on a Reading Comprehension Test (Age 12+) of 1000 students are given below :

Scores (out of 75)	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50
Frequency	6	12	50	120	225	250	185	110	32	10

(*a*) Find the mean score.

(*b*) Find also the median score.

(*c*) Use the formula for Empirical Relation between the mean, median and mode to calculate the mode of the above data.

18. The following table gives the population of males and females in different age-groups according to the 1951 Census of India.

Age-group (in years)	5–14	15–24	25–34	35–44	45–54	55–64	65–74
Number of Males (in lakhs)	447	307	279	220	157	91	39
Number of Females (in lakhs)	420	300	206	205	133	86	40

(*a*) Calculate the average age of the males and of the females.

(*b*) Calculate also their median ages.

19. The class-marks of the distribution of scores in a final examination are 10, 30, 50, 70, 90. Determine the class size and class limits.

20. The class marks of a distribution are

47, 52, 57, 62, 67, 72, 77, 82

Determine the class size and the class limits.

23.5 The Arithmetical Descriptors (or Measures) of Dispersion

Two sections of 10 students each in Class VI in a certain school were given a common test in English (40 maximum marks).

The scores of various students are given below :

Section I. 7, 10, 12, 13, 15, 20, 21, 28, 29, 35

Section II : 15, 15, 15, 15, 18, 19, 21, 22, 25, 25

The average score in each section is 19. Let us construct a dot-diagram, on the same scale, for each distribution (See Fig 23.1 below.) The position of the mean is marked by an arrow.

Performance of students in Section I :

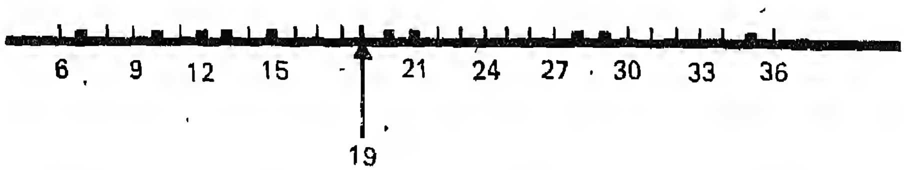


Fig. 23.1 (*i*)

Performance of students in Section II :

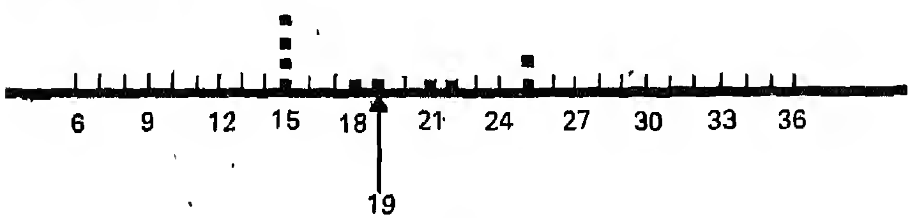


Fig. 23.1 (*ii*)

Clearly, the extent of spread or dispersion is different in each section. The range of scores in Section I is 28 while in Section II is 10.

Now, what should a suitable descriptor of dispersion be ? Examining the two dot-diagrams above, we note that the scores in Section II *cluster around the mean* while those in Section I are *spread farther away from the mean.* This gives us a clue for a possible descriptor as follows :

Let us take the distance (or deviation) of each observation from the mean and add all such deviations. If the sum is '*large*' the dispersion is '*large*'. If, however, the sum is '*small*', the dispersion is '*small*'

Let us find the sum of deviations from the mean for scores in Section I

Observations x_i	*Deviations* $x_i - \bar{x}$
7	−12
10	− 9
12	− 7
13	− 6
15	− 4
20	+ 1
21	+ 2
28	+ 9
29	+10
35	+16
190	0

Unfortunately, the sum is zero. It is neither *'large'* nor *'small'*. Is it a coincidence ? Let us find the sum of deviations from the mean for scores in Section II.

Observations x_i	*Deviations* $x_i - \bar{x}$
15	−4
15	−4
15	−4
15	−4
18	−1
19	0
21	+2
22	+3
25	+6
25	+6
190	0

Again, the sum is zero. Certainly it is no coincidence. In fact, we have proved in earlier classes, that the sum of the deviations from the mean is always zero for any set of data. Why is the sum always zero ?

On close examination, we find that the signs of some deviations are positive and of some other deviations are negative. Perhaps, this is what makes their sum always zero. Now, how do we get rid of negative signs ?

One method is to take the absolute value. If we follow this method, we will obtain a descriptor called the **mean deviation from the mean.**

Another method is to square a number. Let us, therefore, square each deviation (from the mean) and sum We obtain the **sum of squared deviations.** If we divide this sum by the number of observations, we obtain the **average (of) squared deviations** or **variance.** This descriptor, of course, will have **units which are squares of those in the data.**

If we wish to have a descriptor of dispersion with the same units as that of the data, all we need to do is take the positive square root of variance. We obtain the **root mean squared deviation or standard deviation.**

The other descriptors of dispersion are the mean deviation from the median, mean deviation from the mode, quartile deviation, etc. We shall not take them up in this unit.

23.6 Standard Deviation and Variance* of Raw Data

If there are n observations, $x_1, x_2, x_3, \ldots, x_n$, then

$$\text{Variance} = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n}$$

Variance is usually denoted by s^2.

Thus, $$s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$$

The standard deviation,** denoted by s, is, of course, the positive square root of s^2.

Thus, $$s = +\sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}}$$

*There are several names in vogue for variance. For instance, the average squared deviation, the mean squared deviation, etc.

**Another name for standard deviation is root mean squared deviation.

The following steps are employed to calculate the variance and, hence, the standard deviation of raw data. The mean is assumed to have been calculated already.

Step 1 : Make a column of deviations from the mean, namely, $x_i - \bar{x}$

Step 2 (Check) : Sum of deviations must be zero, *i.e.*,

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0$$

Step 3 : Square each deviation and write in the column headed $(x_i - \bar{x})^2$

Step 4 : Find the sum of the column in Step 3.

Step 5 : Divide the sum obtained in Step 4 by the number of observations n. We obtain s^2.

Step 6 · Take the positive square root of s^2. We obtain s.

Example 1 The daily sale of sugar in a certain grocery shop is given below :

Monday	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*
75 kg	120 kg	12 kg	50 kg	70 5 kg	140 5 kg

The average daily sale is 78 kg. Calculate the variance and the standard deviation of the above data.

Solution : We go through the six steps outlined above in the following columnar work :

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
75	— 3	9
120	42	1764
12	—66	4356
50	—28	784
70.5	— 7.5	56.25
140.5	62 5	3906.25
Check	——→0	10875.50

Thus, $$s^2 = \frac{10875.50}{6} = 1812\ 58 \text{ (approx.)}$$

And $$s = \sqrt{1812.58} = 42\ 57 \text{ (approx.)}$$

Example 2 : Refer to the scores (out of 40) of 10 students of Section 1 in a test in English. (See Section 23 5)

7, 10, 12, 13, 15, 20, 21, 28, 29, 35

The mean score is 19. Determine the variance and the standard deviation.

Solution : The various steps are employed in columnar calculations below :

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
7	-12	144
10	-9	81
12	-7	49
13	-6	36
15	-4	16
20	$+1$	1
21	$+2$	4
28	$+9$	81
29	$+10$	100
35	$+16$	256
Check	$\longrightarrow 0$	768

Hence, $s^2 = \dfrac{768}{10} = 76.8$

And, $s = \sqrt{76.8} = 8.76$ (approx.)

23.7 Standard Deviation and Variance of Raw Data—an Alternative Method

If $\bar{x}$ is in decimals, taking deviations from $\bar{x}$ and squaring each deviation involves even more decimals and the computations become tedious. We give below an alternative formula for computing s^2 In this formula, we bypass the calculation of $\bar{x}$

$$s^2 = \frac{\sum_{i=1}^{n} x_i^2 - \dfrac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

Of course, $s = +\sqrt{s^2}$

The steps to be employed in calculation of s^2 and, hence, s by this method are :

Step 1 : Make a column of squares of observations, *i.e.*, x_i^2

Step 2 : Find the sum of the column in Step 1. We obtain $\sum_{i=1}^{n} x_i^2$

Step 3 : Substitute the values of $\sum_{i=1}^{n} x_i^2$, n and $\sum_{i=1}^{n} x_i$ in the above formula. We obtain s^2.

Step 4 : Take the positive square root of s^2. We obtain s

We refer to Example 2 in Section 23 6 and re-calculate the variance and the standard deviation by this method

x_i	x_i^2
7	49
10	100
12	144
13	169
15	225
20	400
21	441
28	784
29	841
35	1225
190	4378

$$s^2=\frac{\sum_{i=1}^{n} x_i^2-\frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

$$=\frac{4378-\frac{(190)^2}{10}}{10}=\frac{768}{10}=76.8$$

Whence, $s=\sqrt{76.8}=8.76$ (approx.)

We observe that we get the same value of s^2 and, hence, s by either method.

23.8 Standard Deviation and Variance of Grouped Data—Method I

Here again we are forced to make an assumption—the same we made in the calculation of the mean of grouped data namely,

Frequencies in each class are centred at its class mark.

We are given k classes and their corresponding frequencies. We will denote the variance and the standard deviation of grouped data by s_g^2 and s_g respectively. The formulae are given below :

$$s_g^2=\frac{\sum_{i=1}^{k} (y_i-\bar{y})^2 f_i}{n}$$

and $$s_g=+\sqrt{s_g^2}$$

The following steps are employed to calculate s_g^2 and, hence, s_g. The mean is assumed to have been calculated already.

Step 1 : Make a column of class marks of the given classes, namely, y_i

Step 2 : Make a column of deviations of class marks from the mean, namely, $y_i-\bar{y}$. Of course, the sum of these deviations need not be zero, since y_i's are no more the original observations.

Step 3 : Make a column of squares of deviations obtained in Step 2, *i.e.*, $(y_i - \bar{y})^2$

Step 4 : Multiply each entry in Step 3 by the corresponding frequency. We obtain $(y_i - \bar{y})^2 f_i$

Step 5 : Find the sum of the column in Step 4 We obtain $\sum_{i=1}^{k} (y_i - \bar{y})^2 f_i$

Step 6 : Divide the sum obtained in Step 5 by n We obtain s_g^2

Step 7 : $s_g = +\sqrt{s_g^2}$

We consider an example.

Example : In a study to test the effectiveness of a new variety of wheat, an experiment was performed with 50 experimental fields and the following results were obtained :

Yield per Hectare (in quintals)	*Number of Fields*
31—35	2
36—40	3
41—45	
46—50	12
51—55	16
56—60	5
61—65	2
66—70	2

The mean yield per hectare is 50 quintals Determine the variance and the standard deviation of the above distribution.

Solution : The various steps are employed in columnar calculations below :

Yield per Hectare (in quintals)	*Number of Fields* (f_i)	*Class mark* (y_i)	$(y_i - \bar{y})$	$(y_i - \bar{y})^2$	$(y_i - \bar{y})^2 f_i$
31—35	2	33	−17	289	578
36—40	3	38	−12	144	432
41—45	8	43	− 7	49	392
46—50	12	48	− 2	4	48
51—55	16	53	+ 3	9	144
56—60	5	58	+ 8	64	320
61—65	2	63	+13	169	338
66—70	2	68	+18	324	648
TOTAL	50				2900

Thus, $$s_g^2 = \frac{\sum_{i=1}^{k} (y_i - \bar{y})^2 f_i}{n} = \frac{2900}{50} = 58$$

and $$s_g = \sqrt{58} = 7.61 \text{ (approx.)}$$

23.9 Standard Deviation and Variance of Grouped Data—Method II

If $\bar{y}$ is not given or if $\bar{y}$ is in decimals in which case the calculations become rather tedious, we employ the alternative formula for the calculation of s_g^2 as given below :

$$s_g^2 = \frac{\sum_{i=1}^{k} y_i^2 f_i - \frac{\left(\sum_{i=1}^{k} y_i f_i\right)^2}{n}}{n}$$

and $$s_g = +\sqrt{s_g^2}$$

The following steps are employed in calculating s_g^2, and, hence, s_g by this method ·

Step 1 : Make a column of class marks of the given classes, namely, y_i

Step 2 : Find the product of each class mark with the corresponding frequency Write the product in the column headed $y_i f_i$

Step 3 : Sum the entries obtained in Step 2. We obtain, $\sum_{i=1}^{k} y_i f_i$

Step 4 : Make a column of squares of the class marks of the given classes namely, y_i^2.

Step 5 : Find the product of each entry in the column of Step 4 with the corresponding frequency. We obtain $y_i^2 f_i$.

Step 6 : Find the sum of the entries obtained in Step 5 We obtain $\sum_{i=1}^{k} y_i^2 f_i$

Step 7 : Substitute the values of $\sum_{i=1}^{k} y_i^2 f_i$, n and $\sum_{i=1}^{k} y_i f_i$ in the formula obtain s_g^2

Step 8 : $s_g = +\sqrt{s_g^2}$.

We find s_g^2 and s_g for the Example of Section 23.8 by this method. The calculations are given below :

Yield per Hectare (in quintals)	f_i	y_i	$f_i y_i$	y_i^2	$y_i^2 f_i$
31-35	2	33	66	1089	2178
36-40	3	38	114	1444	4332
41-45	8	43	344	1849	14792
46-50	12	48	576	2304	27648
51-55	16	53	848	2809	44944
56-60	5	58	290	3364	16820
61-65	2	63	126	3969	7938
66-70	2	68	136	4624	9248
TOTAL	50		2500		127900

Substituting the values of $\sum_{i=1}^{k} y_i^2 f_i$, n and $\sum_{i=1}^{k} y_i f_i$ in the formula, we obtain

$$s_g^2 = \frac{127900 - \frac{(2500)^2}{50}}{50} = \frac{2900}{50} = 58$$

and $s_g = \sqrt{58} = 7.61$ (approx)

Again, we observe that we get the same value of s_g^2 and, hence, s_g by either method.

As we noted in case of grouped-data mean and raw-data mean that the two are, in general, not equal, we would also not, in general, expect s_g^2 to equal s^2 and hence, s_g to equal s. In other words, in general,

$$s_g^2 \neq s^2 \text{ and } s_g \neq s$$

23.10 Standard Deviation and Variance Again : Step Deviation Method

We will see how to simplify calculations, as we did in the case of the mean, by using the step deviation method. We let

$$u_i = \frac{y_i - a}{c} \tag{1}$$

Then, $\quad y_i = cu_i + a \quad$ (2)

We have already shown that

$$M_g \text{ (or } \bar{y}) = c\,\bar{u} + a \tag{3}$$

Subtracting (3) from (2), we get

$$y_i - \bar{y} = c(u_i - \bar{u}) \tag{4}$$

In (4), squaring both sides, multiplying by f_i and summing over k classes, we get

$$\sum_{i=1}^{k} (y_i - \bar{y})^2 f_i = \sum_{i=1}^{k} c^2 (u_i - \bar{u})^2 f_i$$

i.e.,

$$\sum_{i=1}^{k} (y_i - \bar{y})^2 f_i = c^2 \sum_{i=1}^{k} (u_i - \bar{u})^2 f_i \tag{5}$$

Dividing both sides in (5) by n, we have

$$\frac{\sum_{i=1}^{k} (y_i - \bar{y})^2 f_i}{n} = \frac{c^2}{n} \sum_{i=1}^{k} (u_i - \bar{u})^2 f_i$$

i.e.,

$$s_y^2 = c^2 s_u^2 \tag{6}$$

where s_y^2 is the variance of the original data and s_u^2 is the variance of the coded data or, **coded variance.** s_u^2 can be calculated by using the formula which involves the mean, namely,

$$s_u^2 = \frac{1}{n} \sum_{i=1}^{k} (u_i - \bar{u})^2 f_i \tag{7}$$

or by using* the formula which does not involve the mean, namely,

$$s_u^2 = \frac{\sum_{i=1}^{k} u_i^2 f_i - \frac{\left(\sum_{i=1}^{k} u_i f_i\right)^2}{n}}{n} \tag{8}$$

Of course, it is trivial to see that

$$s_y = c\, s_u \tag{9}$$

We consider an example.

Example : We refer to the Example of Section 23 8 again and find the variance using the coded variance.

The various steps are shown in columnar calculations below. We let $a=48$. We know $c=5$.

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* Since we do not know, before we start doing our calculations, whether $\bar{u}$ will be a 'nice' number or not, we prefer to use formula (8) in our calculations.

Yield per Hectare (in quintals)	Number of Fields (f_i)	Class mark (y_i)	$u_i=\frac{y_i-48}{5}$	f_iu_i	u_i^2	$f_iu_i^2$
31—35	2	33	−3	−6	9	18
36—40	3	38	−2	−6	4	12
41—45	8	43	−1	−8	1	8
46—50	12	48	0	0	0	0
51—55	16	53	+1	16	1	16
56—60	5	58	+2	10	4	20
61—65	2	63	+3	6	9	18
66—70	2	68	+4	8	16	32
TOTAL	50			20		124

Thus $$s_u^2=\frac{\Sigma f_iu_i^2-\frac{(\Sigma f_iu_i)^2}{n}}{n}$$

$$=\frac{124-\frac{(20)^2}{50}}{50}=\frac{58}{25}$$

We now use (6) and get the variance of the original data as

$$s_y^2=c^2s_u^2$$

i.e., $s_y^2=(5)^2\,\frac{58}{25}=58$

We, of course, get the same variance (and hence, standard deviation) as before.

23.11 Review of the Formulae and an Aid to Memory

Raw Data

Given : Observations :

$$x_1, x_2, x_3, \ldots, x_n$$

Number of observations : n

Grouped Data

Given : Class marks :

$$y_1, y_2, y_3, \ldots, y^k$$

and frequencies : $f_1, f_2, f_3, \ldots, f_k$

Number of observations : n

Number of classes : k

Class size : c

$$f_1+f_2+\ldots+f_k=\sum_{i=1}^{k} f_i=n$$

Mean : M or $\bar{x}$

$$M=\frac{\sum_{i=1}^{n} x_i}{n}$$

Mean : M_g or $\bar{y}$

Assumption : Frequencies in each class are centred at its class mark

$$M_g \text{ (or } \bar{y})=\frac{\sum_{i=1}^{k} y_i f_i}{n}$$

Mean by coding (u-scale) :

$$y_i=cu_i+a$$

$$M_g \text{ (or } \bar{y})=cM_u+a$$

where $M_u=\dfrac{\sum_{i=1}^{k} u_i f_i}{n}$

Median :

If n is odd, value of $\left(\frac{n+1}{2}\right)$th observation.

If n is even, average of the values of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

Median :

Assumption : Frequencies in each class are distributed uniformly (evenly) throughout the class.

$$\text{Med}=L_{\text{med}}+\frac{j\times c}{f_{\text{med}}}$$

where

L_{med} =Lower limit of the median-class,

$j=\frac{n}{2}-c.f.$ of Med_{-1} class

f_{med} =Frequency of the median-class

Mode :

Value of the observation that occurs most often

Mode :

For a 'crude' calculation,

Assumption : Frequencies in each class are centred at its class mark

Mode=Class mark of the modal-class

$$\text{Mode} = \text{Mean} - 3\,(\text{Mean} - \text{Median})$$

Variance : s^2

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

Formula without the use of mean :

$$s^2 = \frac{\sum_{i=1}^{n} x_i^2 - \dfrac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

Standard deviation : s

$$s = +\sqrt{s^2}$$

Variance : s_g^2

Assumption : Frequencies in each class are centred at its class mark

$$s_g^2 = \frac{\sum_{i=1}^{k} (y_i - \bar{y})^2 f_i}{n}$$

Formula without the use of mean :

$$s_g^2 = \frac{\sum_{i=1}^{k} y_i^2 f_i - \dfrac{\left(\sum_{i=1}^{k} y_i f_i\right)^2}{n}}{n}$$

Variance by coding (u-scale) :

$$y_i = c u_i + a$$

$$s_y^2 = c^2 s_u^2$$

where

$$s_u^2 = \frac{\sum_{i=1}^{k} (u_i - \bar{u})^2 f_i}{n}$$

$$\text{or, } s_u^2 = \frac{\sum_{i=1}^{k} u_i^2 f_i - \dfrac{\left(\sum_{i=1}^{k} u_i f_i\right)^2}{n}}{n}$$

Standard deviation : s_g

$$s_g = +\sqrt{s_g^2}$$

Standard deviation by coding (u-scale) :

$$y_i = c u_i + a$$

$$s_y = c s_u$$

where $s_u = +\sqrt{s_u^2}$

The formula for variance of raw data is easy to write if we recall that s^2 is also called the mean squared deviation. Thus,

(*i*) We write a deviation from the mean, namely, $x_i - \bar{x}$

(*ii*) We square it, namely, $(x_i - \bar{x})^2$

(*iii*) We find the average, namely, we add and divide by the total number and obtain

$$\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$ which is, of course, the formula for s^2.

The second formula for s^2 is mathematically equivalent to the first one.

Now how about the formulae for grouped data ? If we can write down a formula for raw data, the corresponding formula for grouped data is easy to write Below are the corresponding changes to be made .

(*i*) Instead of observations x_i's, class marks y_i's are to be written.

(*ii*) Class marks, by themselves, are meaningless. Thus, they have to be multiplied by the corresponding frequencies

(*iii*) Summation extends over k classes rather than the number of observations.

EXERCISE 23.2

1. Total daily evaporation (in mm) for Delhi for the month of July 1973, as reported by the Meteorological Department is given below. Find the standard deviation .

   12 0, 16.2, 12 0, 14.0, 6 9, 4.2, 9.0, 14 0, 14.0, 11 5, 11.5, 14.5, 17.0, 4 4, 9.7, 4.9, 8.4, 5.3, 9.2, 6.5, 0.0, 7.2, 9 0, 4.0, 7.1, 3.0, 8.1, 8.0, 5.6, 8.1, 4.7

2. Durations of sunshine (in hours) in Amritsar for the first ten days of August 1974, as reported by the Meteorological Department are given below :
   5.1, 4 7, 3.1, 1 6, 1.7, 5 4, 5.0, 11.7, 11 6, 11.5
   Calculate s^2 and s.

3. Given below is a part of the Fibonacci sequence :

   1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89

   Find s^2 and s

4. Find the variance of first n natural numbers.

5. In a study on diabetic patients, the following data are obtained. Find the average age at first detection as also the variance of the data.

Age at Detection (in years)	10—19	20—29	30—39	40—49	50—59	60—69	70—79	80—89	TOTAL
Number of Cases	1	0	1	10	17	38	9	3	79

6. Refer to Question 4, Exercise 23.1 Find s_g^2 and s_g

7. The daily earnings of 30 drug stores are given below :

Daily Earnings (in Rs)	Up to 50	51—100	101—150	151—200	201—250	251—300	301—350	351—400	TOTAL
Number of Stores	3	7	4	5	4	3	2	2	30

) Draw a histogram of the above distribution.

Calculate M_g and s_g.

(c) Give an interpretation of M_g.

(d) Mark the position of the mean on the histogram.

(e) Calculate the following intervals .

(i) $(M_g - s_g, M_g + s_g)$

(ii) $(M_g - 2s_g, M_g + 2s_g)$

(iii) $(M_g - 3s_g, M_g + 3s_g)$

8. The maximum temperature (in °C) for the first ten days of June 1978 in a certain city are

43, 44, 42, 41, 36, 43, 42, 40, 39, 40

(a) Compute M_g and s_g^2

(b) If each of the temperatures is increased by 2 °C, find the mean and variance of the new set of data

9. The variance of 8 observations is 2 5. If each observation is multiplied by 4, find the variance of the new set of data.

10. Following are the weights (in kg) of 20 new-born babies in a maternity home in Chandigarh on a particular day in the month of October 74 :

3.0, 3.5, 3.0, 2.5, 2.75, 2 0, 2 25, 2 25, 2.0, 2.2, 2.75, 2.0, 3.2, 3.5, 4.0, 3.5, 3.2, 2.0, 2 5, 2 0

(a) Calculate M, s^2 and s

(b) Construct a frequency table using a class size of 0.2

(c) Draw a histogram to represent the above data

(d) Calculate M_g, s_g^2 and s_g

(e) Is $M=M_g$?

(f) Is $s^2=s_g^2$?

(g) Is $s=s_g$?

11. Compute the mean and standard deviation of the following scores :

7, 15, 10, 9, 4, 3, 7, 10, 10, 8, 16

12. The data below give the earnings of 432 workers in a flour mill

Monthly Wages (in rupees)	*Number of Workers*
80-100	18
100-120	30
120-140	20
140-160	40
160-180	90
180-200	70
200-220	68
220-240	36
240-260	27
260-280	21
280-300	12

Calculate the average earnings of the group and the standard deviation.

13. The marks obtained (out of 800) by 1200 students in an entrance examination are given below .

Marks	*Number of Students*
201-250	62
251-300	120
301-350	412
351-400	100
401-450	379
451-500	87
501-550	23
551-600	12
601-650	3
651-700	2

Refer to the above frequency distribution and write the value of

(*i*) k, the number of classes

(*ii*) y_3, class mark of the 3rd class.

(*iii*) f_5, frequency of the 5th class

Also, calculate the mean, standard deviation and variance.

14. Following is the distribution of pupils attending secondary schools and the distance of a pupil's home from a school as reported in the Second All India Educational Survey in rural areas of Pondicherry

Distance (*in miles*)	*Pupils Attending*
Up to 1.0	540
1.1 to 2.0	478
2 1 to 3.0	434
3.1 to 4.0	81
4.1 to 5 0	113
5.1 to 6 0	30

(*a*) Calculate M_g, s_g^2 and s_g

(*b*) Draw a histogram of the above distribution and mark the position of the mean.

(*c*) Calculate the following intervals :

(*i*) $(M_g - s_g, M_g + s_g)$

(*ii*) $(M_g - 2s_g, M_g + 2s_g)$

(*iii*) $(M_g - 3s_g, M_g + 3s_g)$

(*d*) Mark the positions of M_g-3s_g and M_g+3s_g on the histogram.

(*e*) Approximately how many observations lie in the interval (M_g-3s_g, M_g+3s_g) ? In the interval (M_g-2s_g, M_g+2s_g) ?

23.12 Key Concepts

Raw data	Empirical relation between mean, median and mode
Grouped data	Variance
Mean	Variance by coding
Mean by coding	Standard deviation
Median	Standard deviation by coding
Mode	

23.13 Suggestions For Further Reading

An excellent book for a serious student of statistics is

[1] G.U. Yule and M.G. Kendall : **An Introduction to the Theory of Statistics.**
Charles Griffin and Company, Ltd., London (U.K.). 1958

Problems from various fields of application and methods to decide which statistical tool to use, how to use it and how to interpret the conclusions, are elegantly presented in

[2] B. Ostle : **Statistics in Research,** Revised Second Edition.
The Iowa State University Press, Iowa (U S.A.). 1972

Statistical tools in the hands of an inexperienced researcher can be dangerous. The reader is referred to the paperbacks

[3] W.J. Reichmann : **Use and Abuse of Statistics.**
Penguin Books, Middlesex (U K.). 1976.

[4] D. Huff . **How to Lie with Statistics.**
W.W. Norton and Company, Inc , New York (U.S.A.). 1954.

A comprehensive reading on the subject is the paperback of long standing

[5] M.J Moroney : **Facts From Figures.**
Penguin Books, Middlesex (U.K). 1958

ANSWERS

UNIT XVII

Exercise 17.1

8. Finds the smallest of the three given numbers.

9. Counts the numbers which are greater than or equal to 10 from among a given set of numbers.

UNIT XVIII

Exercise 18.1

1. (*i*) 4 units/sec
   (*ii*) 7 units/sec
   (*iii*) 8 units/sec

Exercise 18.2

1. (*a*) 15.6
   (*b*) 15.8
   (*c*) 16 2
   (*d*) 16 6
2. 2, 6, 8, No
3. 3.2 metres/sec
4. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
5. $-4x$, 0, 4, -6
6. $2\pi r$, 5π, $2\pi^2$
7. 2
8. $6.4t$
9. (*a*) 1
   (*b*) 0

UNIT XIX

Exercise 19.1

1. $-\frac{1}{2}$
2. 7
3. $6x$
4. $-\frac{4}{5}x$
5. $-18x^2$
6. $32x^3$
7. $\frac{35}{2}x^6$
8. $-3x^7$
9. 0
10. 0
11. 16, 16, 16
12. 9 8, 49

13. $-\frac{1}{6561}$, 0, 1536

14. 0, 5, 80, 405

15. $-\frac{1}{4}, \frac{1}{4}$

16. 1

17. $4\pi r^2$, 16π

18. $4x^3, -\frac{1}{2}$, 0

Exercise 19.2

2. x^2-4x+6

3. $\frac{a}{c}$

4. t^4-t^2+1

5. $3x^2-12x+8$

6. $18x^5-x^3$

7. $8x-8$

8. $3x^2-6x+3$

9. $12t^3-10t+1$

10. $u^7-u^5+u^3$

11. $-20y^9-3y^2+35y^4$

12. t^3-t+1

13. y^2-2y+1

14. $28t^3-6t^2-8$, -8, 14, 192

15. 1, 1, 3

16. 4

17. 85, -55

18. -6

19. 202

20. $-\frac{19}{9}$, -19, -67

21. (*a*) 19.6 m/sec, 9 8 m/sec
 (*b*) 3 seconds
 (*c*) 44.1 metres

22. 21 m/sec

23. Rs 8400

UNIT XX

Exercise 20.1

1. (*i*) 9.8
 (*ii*) -4
 (*iii*) 0
 (*iv*) 0

2. 3, 0, 3

Exercise 20.2

1. $y-4x+2=0$

2. $y-16=0$

3. $y-256=0$

4. $y+8x+9=0$

5. $y-27x-54=0$

6. (7, 23)

7. $\left(\frac{8}{3}, \frac{128}{27}\right), \left(\frac{-8}{3}, -\frac{128}{27}\right)$

8. (*a*) (0, 1)
 (*b*) (1, 2)

9. (a) $(0, 0), \left(\frac{1}{3}, \frac{1}{27}\right)$

(b) $(0, 0), (3, 27)$

12. $(0, -2)$

14. $4y-12x+65=0$

15. $(1, 4)$

16. $6y+x-56=0$

17. $3y+x-8=0$

18. $y-x=0$

19. $6y-x-30=0$

20. $4y+x-17=0$

21. $9y+x-20=0$, $9y+x+20=0$

23. $\left(\frac{3}{2}, -\frac{17}{2}\right)$

24. $\left(\frac{1}{2}, -\frac{17}{4}\right)$

25. $2y-x-4=0$

UNIT XXI

Exercise 21.1

2. $4x-7$

3. $9x^2+28x+\frac{3}{2}$

4. $8x^3+9x^2+16x$

5. $2x^9+\frac{8\sqrt{2}}{5}x^7+7\sqrt{3}\,x^6+5\sqrt{6}\,x^4$

6. $4acx^3-2(bc+ad)x$

7. $4x^3+8x$

8. $3(x+1)^2$

9. $4x^3-6x^2-6x$

10. $\frac{45}{4}x^4+18x^3+9x^2+12x+1$

11. $35x^6+30x^4-15x^2-6$

Exercise 21.2

1. $\frac{-1}{(x+1)^2}$

2. $\frac{(x+2)(x-4)}{(x-1)^2}$

3. $\frac{-7}{x^8}$

4. $-\frac{\sqrt{3}}{x^4}$

5. $-\frac{(x+10)}{2x^3}$

6. $-\frac{1}{(2x-1)^2}$

7. $-\frac{4}{x^5}+\frac{3}{(x-1)^2}-\frac{10}{x^6}$

8. $4x^3-\frac{1}{x^2}+\frac{16}{x^5}$

9. $\frac{4x^2+12x+2}{(2x+3)^2}$

10. $2x+3+\frac{4}{x^2}-\frac{28}{x^5}$

11. $\frac{4x}{(x^2+1)^2}$

12. $\frac{6x^2+30x+29}{(2x+5)^2}$

13. $\dfrac{x^3+6x^2+16}{(x+2)^3}$

14. $-4x^2+20x-22$

15. $\dfrac{2x^3-6x^2-6}{(x-2)^2}$

16. $\dfrac{4x^3-23x^2-8x-5}{(x-4)^2}$

17. $\dfrac{-1}{(x-a)(x-b)(x-c)} \times \left[\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}\right]$

Exercise 21.3

1. $6(x+1)^5$

2. $-9(2-3x)^2$

3. $12x^2(x^3+2)^3$

4. $10(x+1)(x^2+2x-11)^4$

5. $-\dfrac{3}{(x-1)^4}$

6. $-4(4x+5)(2x^2+5x-3)^{-5}$

7. $(3x^2+2)^2(5x-1)[10(3x^2+2)+18x(5x-1)]$

8. $4x(x^2+3)^3(x^2+5)(3x^2+15)$

9. $\dfrac{12(2t^3+1)(t^4+t^2-t)}{(3t^2+1)^3}$

10. $\dfrac{-2(12x-9)}{(4x+3)^3(2x-3)^2}$

11. $\dfrac{4(x^3-2x)^3(2x^4+21x^2-14)}{(x^2+7)^3}$

12. $-\dfrac{24t}{(2t^2+5)^3}$

13 $\dfrac{3}{(2-x)^2}$

14. $\dfrac{50x(3x-8)}{(3x+2)^4}$

15 $3x^2+12x+11$

16. $\dfrac{2}{27}\,x^2(2x^3+15)^3$

17. $\dfrac{-5(x^2+1)}{(1+2x-x^2)^2}$

18 $-54x+81x^3$

19 $(ax+b)^{m-1}\,(cx+d)^{n-1} \times [cn(ax+b)+am(cx+d)]$

20. t^3+5t

21 $\dfrac{27\pi}{8}(2x+3)^3$

Exercise 21.4

1. $-\dfrac{3x^2+8y}{8x+3y^2}$

2. $-\dfrac{x^5+2xy^2}{y^5+2x^2y}$

3. $-\dfrac{ax+hy+g}{hx+by+f}$

4. $-\dfrac{16x}{9y}$

5. $\dfrac{3-x}{y-5}$

6. $\dfrac{15x^2-72xy+54y^2}{36x^2-108xy+81y^2-2y}$

7. $2y+5x-9=0$

8. $y=0$

9. $x=2$

10. $-\frac{8}{3}\ x^{-\frac{11}{8}}$

11. $x^{-\frac{1}{4}}$

12. $\frac{1}{2\sqrt{x}}+\frac{1}{2x^{\frac{3}{2}}}$

13. $\frac{8x}{5(4x^2+5)^{\frac{4}{5}}}$

14. $\frac{a}{3}\ (ax+b)^{-\frac{2}{3}}$

15. $\frac{-40x^6+60x^5-14x^3+123x^2-80x+8}{2(2x^3-5x^2+8)^5(x^4+x-1)^{\frac{1}{2}}}$

16. $\frac{1}{3}(15x^2-47x+36)^{-\frac{2}{3}}(30x-47)$

17. $\frac{-3}{2(x+2)^{\frac{3}{2}}\ (1-x)^{\frac{1}{2}}}$

18. $\frac{(11x^5+2x-4)\sqrt{x^5-2x+1}}{2x^3}$

MISCELLANEOUS EXERCISE V

(On Units XVIII, XIX, XX, XXI)

1. 3640 units/sec

2. (i) $\frac{3(t^6-1)}{t^4}$, $\frac{728}{27}$

(ii) $t^2(t^3+10)^{-\frac{2}{3}}$, $9(37)^{-\frac{2}{3}}$

3. 9

4. $20x-\frac{15}{x^2}$ Approx. Rs 2000

5. $21-\frac{2x}{5}$, Rs 11

6. $1+4x-3x^2$

7. $x^6+x^4-2x^2$

8. $40x^7-18x^2+2x$

9. $\frac{4}{15}\ x^{-\frac{1}{3}}+\frac{4}{5}\ x^{-\frac{9}{5}}-6x^{-3}$

10. $\frac{3}{(1-3x)^2}-\frac{2}{5}\ x^{-\frac{3}{5}}$

11. $\frac{1}{2}\left(\frac{1}{\sqrt{x}}-\frac{1}{x^{\frac{3}{2}}}\right)\left(1+x+x^2\right)$

$+\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(1+2x\right)$

12 $(x^2-2)(6x^3+15x^2+16x-6)$

13. $\frac{2(x-\sqrt{x})}{(1-2x)^3}\left[1-\sqrt{x}-\frac{1}{2\sqrt{x}}\right]$

14. $\frac{x^4-4x^3-5x^2-2x}{(x^3+x^2)^2}+\frac{1}{3}\ x^{-\frac{2}{3}}$

$+\frac{1}{3}\ x^{-\frac{4}{3}}$

15. $\frac{3}{2}\ (1-2x^2)^2(1-x)^{\frac{1}{2}}\ (10x^2-8x-1)$

16. $1-\frac{1}{(x-1)^2}-\frac{x}{\sqrt{100-x^2}}$

17. $\frac{9-2x^2}{\sqrt{9-x^2}}+\frac{6-3x}{2\sqrt{4-x}}$

18. $9\pi r^2+20\pi r+8\pi$

19. $-4\pi r^3+3(\pi-1)r^2+2r$

20. $-\frac{4\pi^2}{r^5}-2r$

21. $\dfrac{5x^4+6x}{3(x^5+3x^2)^{\frac{2}{3}}}$

22. $1+\dfrac{x}{\sqrt{x^2+8}}$

23. $\dfrac{-x^2-1}{(x^2-x-1)^2}$

24. $\dfrac{x^2-1}{x^2}$

25. $\dfrac{x}{2a}$

26. $3+\dfrac{1}{\sqrt{2}}$

27. $\dfrac{3x^2-6xy-1}{3(x^2+1)}$

28. $-\sqrt{\dfrac{y}{x}}$

29. $\dfrac{-(3yx^2+4x^3+3)}{(2+x^3)}$

30. $\dfrac{y^2}{x^2}$

31 $\dfrac{ay-x^2}{y^2-ax}$

32. 4

33. -1

34. 0

35. $y-7x+10=0$

36. $3y-4x+25=0$

37. $x+y-2=0$

38. $9y-x+27=0$

39. (3, 2)

40. (4, 5)

41. (*a*) $40y-80x+103=0$

(*b*) $2\sqrt{2}y-5x+4\sqrt{2}+1=0$

42. $y-12x-38=0$

43. $y+x-1=0$

44. $\sqrt{3}y-x=0$

45. $9y+x-55=0$, $9y+x-35=0$

UNIT XXII

Exercise 22.1

1. 8 fans, 12 sewing machines
2. $P=16$
3. $C=0$
4. 4 units of A, 4 units of B ; Rs 40.00
5. A : 5 days, B : 3 days
6. 30 packages of screws 'A' and 20 packages of screws 'B' ; Rs 41.00
7. 4 pedestal lamps, 4 wooden shades
8. Number of quintals to be transported

From

		A	B
	D	10	50
To	E	50	0
	F	40	0

9. Delivery of petrol

From (in litres)

		A	B
	D	500	4000
To	E	3000	0
	F	3500	0

UNIT XXIII

Exercise 23.1

2. $\frac{n+1}{2}$

3. $\frac{(n+1)(2n+1)}{6}$

4. (i) 270.78

5. (i) Rs 21.82

   (ii) Rs 21.54 (Median)

6. 405.45 million km

7. 57.25 kg

8. 7337 1 million hectares, 5981.5 million hectares, 3270.3 million hectares (using the Empirical Relation)

9. 3.9 kg

10. 0.215 sec

11. Rs 248

12. 3 24 (Median)

13. 10

14. It is not possible to find the median of 25 observations with the given information

15. —4

16. 12

17. Mean=Median=Mode=26.74

18. (a) 27.95 years, 27.70 years

    (b) 25.57 years, 24.17 years

19. 20 ; 0-20, 20-40, 40-60, 60-80, 80-100

20. 5 ; 44.5—49.5, 49.5—54.5, 54.5—59.5, 59 5—64.5, 64.5—69 5, 69 5—74.5, 74 5—79.5, 79.5—84 5

Exercise 23.2

1. 4.08

2. 14.40, 3.8

3. 720.26, 26.84

4. $\frac{n^2-1}{12}$

5. 60.7, 127.37

6. 4019.37, 63.4

7. (b) 170.5, 101.94

   (e) (i) (68.56,272.44),

   (ii) (—33.38,374.38),

   (iii) (—135.32,476.32)

8. (*a*) 41,5 ; (*b*) 43,5

9. 40

10. (*a*) 2 7, .3740, .612 ; (*d*) 2.76, .3604, 60, (*e*) No, (*f*) No, (*g*) No

11. 9, 3.79

12. 186.53, 47 51

13 (*i*) 10, (*ii*) 325.5, (*iii*) 379 ; $M_g=369.6$, $s_g=75.09$, $s_g^2=5638.4$

14. (*a*) 1.86, 1.5732, 1.2542

(*c*) (*i*) (.61, 3.11), (*ii*) (− 64, 4.36)

(*iii*) (−1 89, 5.61)

(*e*) 1664, 1578

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